Mathematical Reviews

Edited by

O. Neugebauer J. D. Tamarkin O. Veblen

W. Feller, *Executive Editor*

Vol. 6, No. 4 April, 1945 pp. 85–112

TABLE OF CONTENTS

THE MICROFILM SERVICE

Through the services of the Brown University Photographic Laboratory one can obtain, either on microfilm or as photoprint, the complete text of any article listed in Mathematical Reviews or available in the Brown University Library, except books and material of which the reproduction is prohibited by copyright laws.

Procedure for Ordering Microfilm or Photoprint

1. Give your name and address.

2. State whether you wish microfilm or photoprint.

3. (a) If the article in question has been listed in Mathematical Reviews, give the MF number printed at the end of the title. Titles without MF numbers are copyrighted.

   (b) If the article has not been reviewed (for instance because published prior to 1939) give as accurately as possible the author, title, journal, volume, year and pages.

4. Enclose cash for the amount of the order. U. S. currency or check, made payable to Brown University, will be accepted. The minimum charge is 45¢.

5. Mail to: *Brown University Photographic Laboratory, Providence, Rhode Island.*

Cost

MICROFILMS are furnished for

15¢ for articles with less than 25 pages.

30¢ for articles with more than 24 pages, but less than 50 pages.

50¢ for articles with more than 49 pages, but less than 100 pages.

$1.00 for articles with more than 99 pages.

PHOTOPRINTS cost 20¢ per sheet, each sheet containing two pages of text, except in the case of unusually large journals (larger than 7" by 10"). In the latter case you will be informed of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price of 20¢ per page.

MINIMUM CHARGE. If an order totals less than 45¢ the cash included should be raised to 45¢ in order to cover the basic expenses of this service.

Microfilm and photoprints with MF numbers show black writing on white background. Articles which are not reviewed, and which therefore do not have an MF number, are reproduced only in negative films or prints, i.e. white letters on black background. Microfilms can be read in any reading machine adapted to standard 35 mm. double perforated film. Photoprints are on sheets 10" by 14" (about 25 by 35 cm.).

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of Mathematical Reviews printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained for an additional payment of $1.00. A regular current subscription can be changed to a one-side subscription by paying the additional $1.00. This edition is folded but not stitched.

MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

The American Mathematical Society
The Mathematical Association of America
The Edinburgh Mathematical Society
Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima
Het Wiskundig Genootschap te Amsterdam
The London Mathematical Society
Union Matematica Argentina

Editorial Office

Mathematical Reviews, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to Mathematical Reviews, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Vo

*F
N
x
F
to
Ad
the
the
His
to
a w
nu
abl
and
of
poi
Th
and
Spa
(in
Sch
Ell
Leg
asy
fer
La
T
ter
are

Ca
s
4
[
S
ele
pa

Le
c
[

Bo
c
[

Wa
I

is

Mathematical Reviews

Vol. 6, No. 4 APRIL, 1945 Pages 85-112

ANALYSIS

Calculus

*Franklin, Philip. **Methods of Advanced Calculus.** McGraw-Hill Book Company, Inc., New York, 1944. xii+486 pp. $3.00.

Having given vent to the natural urge of a mathematician to prove all statements in his earlier work [A Treatise on Advanced Calculus, John Wiley and Sons, New York, 1940; these Rev. 2, 77], in the present text the author sets himself the task of stating the facts but omitting involved proofs. His present book is an advanced text for students who want to use mathematics as a tool and as such is crammed with a wealth of material. The first chapter opens with complex numbers and the elementary functions of a complex variable. The notion of a singular point is then developed briefly and the author states the relationship between the radius of convergence of a power series and the location of singular points. The first chapter closes with indeterminate forms. The succeeding chapter headings are: Partial Differentiation and Implicit Functions; Vectors, Curves and Surfaces in Space; Integration; Line Integrals and Complex Variables (including contour integration and the applications of the Schwarz mapping transformation); The Gamma Function; Elliptic Integrals; Vector Analysis; Differential Equations; Legendre Polynomials and Bessel Functions (including asymptotic expansions); Fourier Series and Partial Differential Equations; The Calculus of Variations and LaGrange's Equation.

The exercises contain considerable supplementary material. References to more detailed and complete accounts are given at the end of each chapter. *N. Levinson.*

Caligo, Domenico. Alcuni sviluppi di Taylor considerati sul cerchio di convergenza. Boll. Un. Mat. Ital. (2) 4, 6 pp. (1943) = Ist. Naz. Appl. Calcolo (2) no. 155. [MF 11513]

Si considerano gli sviluppi di Taylor di alcune funzioni elementari sul cerchio di convergenza e se ne deducono particolari sviluppi trigonometrici. *Author's summary.*

~~The paper is quoted from a reprint~~

Levi, B. The approximation-polynomials of $\sin x$ and $\cos x$. Math. Notae 4, 156–163 (1944). (Spanish) [MF 11483]

Boyer, Carl B. Pascal's formula for the sums of powers of the integers. Scripta Math. 9, 237–244 (1943). [MF 11034]

Waterson, A. An expansion for x^n+y^n. Edinburgh Math. Notes no. 34, 14–15 (1944). [MF 11305]

The expansion

$$x^n+y^n=\sum(-1)^r(n/(n-r))\binom{n-r}{r}(x+y)^{n-2r}(xy)^r$$

is proved; the summation extends over $r=0, 1, \cdots, [n/2]$.
W. Feller (Providence, R. I.).

The volume number was quoted from a reprint. The paper actually appeared in v. 5, pp. 168–175

Mordell, L. J. Dirichlet's integrals. Edinburgh Math. Notes no. 34, 15–17 (1944). [MF 11306]

The author gives a simplified rigorous proof of the identity

$$\iiint f(x+y+z)x^{k-1}y^{m-1}z^{n-1}dxdydz$$
$$=\Gamma(k)\Gamma(m)\Gamma(n)\Gamma^{-1}(k+m+n)\int_a^b t^{k+m+n-1}f(t)dt;$$

the triple integral extends over $x, y, z\geqq 0$, $a\leqq x+y+z\leqq b$; moreover, $f(t)$ is supposed to be either continuous or bounded and monotone. *W. Feller* (Providence, R. I.).

Sispánov, Sergio. A maximum problem. Revista Union Mat. Argentina 10, 1–12 (1944). (Spanish) [MF 11027]

The following problem is discussed: to divide a given rectangle into two rectangles, to cut out of one of them the base of a cylindrical container c, and to use the other one as the lateral surface of c, so that the volume of c becomes a maximum. *P. Scherk* (Saskatoon, Sask.).

Sispánov, Sergio. Determination of the volume of a solid. Revista Union Mat. Argentina 10, 37–40 (1944). (Spanish) [MF 11532]

Let P be an elliptic point of the surface S, and let S_α be the surface obtained by taking the surface which is symmetric to S with respect to the tangent plane at P and turning it through α about the normal at P. Generalizing a result of Santaló [same Revista 9, 15 (1943), problem 44] the author shows that the volume $V_\alpha(h)$ between S and S_α satisfies the relation

$$\lim_{h\to 0} h^{-2}V_\alpha(h)=\pi R_1R_2\{4R_1R_2+(R_1-R_2)^2\sin^2\alpha\}^{-1}.$$

P. Scherk (Saskatoon, Sask.).

Differential Equations

Barbachine, E. Les singularités locales des points ordinaires d'un système d'équations différentielles. C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 183–186 (1943). [MF 11062]

This paper concerns systems of ordinary first order differential equations in the standard form, whose left members are continuous but do not necessarily satisfy a Lipschitz condition to ensure uniqueness. In general a whole bundle of solution curves may pass through a given point. The author applies to these bundles Hausdorff's notion of distance between two point sets to define what he calls a point of continuity. He then proves (1) that the set C of all points of continuity contains the set U of all points where the solution is unique and (2) that C is a set of the second category in the sense of Baire. Examples exist to show that

in general U is a proper subset of C. The author makes fundamental use of a result of Kamke [Acta Math. **58**, 57–85 (1932)]. *D. C. Lewis* (New York, N. Y.).

Barbachine, E. Sur certaines singularités qui surviennent dans un système dynamique quand l'unicité est en défaut. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 139–141 (1943). [MF 11057]

It is shown that much of G. D. Birkhoff's theory of recurrent motions of dynamical systems can be generalized to the case when the differential equations do not satisfy a condition to ensure uniqueness. A result of Kamke is, in this connection, of fundamental importance [Acta Math. **58**, 57–85 (1932)]. *D. C. Lewis* (New York, N. Y.).

Caligo, Domenico. Sulle equazioni differenziali lineari del secondo ordine a coefficienti periodici. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 1025–1033 (1942) = Ist. Naz. Appl. Calcolo (2) no. **146**. [MF 11516]

It is stated that G. Calamai [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. 1941] established sufficient conditions for the stability of the solutions of the differential equation

$$y''+p(x)y'+q(x)y=0,$$

where $p(x)$ and $q(x)$ are periodic functions of period $\omega>0$. The present paper improves the conditions imposed by Calamai. In particular, the condition $p(0)=0$ is removed. *F. G. Dressel* (Durham, N. C.).

Wasow, Wolfgang. On the asymptotic solution of boundary value problems for ordinary differential equations containing a parameter. J. Math. Phys. Mass. Inst. Tech. **23**, 173–183 (1944). [MF 11411]

Let $U(x, \lambda)$, $\alpha\leqq x\leqq\beta$, $\lambda>0$, be the solution of the differential system

$$N(y)+\lambda M(y)=\sum_{i=0}^{n} a_i(x)y^{(i)}+\lambda\sum_{j=0}^{m} b_j(x)y^{(j)}=0,$$

$$n>m;\ y^{(i)}=d^iy/dx^i,$$

$$L_i(y)=\begin{cases} y^{(v_i)}(\beta)=l_i, & i=1, \cdots, r, \\ y^{(u_i)}(\alpha)=l_i, & i=r+1, \cdots, n. \end{cases}$$

Here the l_i are constants, and the sets of integers v_i and u_i satisfy the relations $n>v_1>v_2>\cdots>v_r\geqq 0$, $n>u_{r+1}>u_{r+2}>\cdots>u_n\geqq 0$. Under certain conditions of regularity, the author proves that the limiting function $u(x)=\lim_{\lambda\to\infty}U(x, \lambda)$ satisfies the differential equation $M(y)=0$ and a well-defined subset of the boundary conditions $L_i(y)=l_i$. *F. G. Dressel* (Durham, N. C.).

Pfeiffer, G. V. Symbolic forms of the canonical type separating one or a row of linear factors, and the equations linear in Jacobians connected with them. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 285–288 (1944). [MF 11639]

***Bateman, H. Partial Differential Equations of Mathematical Physics.** Dover Publications, New York, N. Y., 1944. xxi+522 pp. $3.95.

This is a new edition of the book which was originally published in 1932 at the Cambridge University Press. While the book is, in the main, a reproduction by the familiar photographic process, not only have known misprints been corrected, but several minor changes and corrections have been introduced by the author.

Bers, Lipman and Gelbart, Abe. On a class of functions defined by partial differential equations. Trans. Amer. Math. Soc. **56**, 67–93 (1944). [MF 10791]

Supplementing a previous paper [Quart. Appl. Math. **1**, 168–188 (1943); these Rev. **5**, 25], the authors investigate from a purely mathematical viewpoint the system of differential equations

$$(*)\qquad \sigma_1(x)u_x=\tau_1(y)v_y,\quad \sigma_2(x)u_y=-\tau_2(y)v_x$$

and show that many concepts and results in the theory of analytic functions of a complex variable can be extended to pairs of functions $u(x, y)$, $v(x, y)$ satisfying the system (*), notably the concepts of differentiation, integration, powers, power series, the theorems of Cauchy and Morera and the fundamental theorem of algebra. The basis for this systematic study seems to be the obvious but important remark that $u=1$, $v=0$ and $u=0$, $v=1$ are solutions of (*), from which powers can be defined by integration in a generalized sense. The assumption $\sigma_1\sigma_2\tau_1\tau_2>0$ is made throughout the paper (elliptic case); however, some of the results hold also for the hyperbolic case $\sigma_1\sigma_2\tau_1\tau_2<0$. *A. Weinstein* (Toronto, Ont.).

Courtel, Robert. Sur la perturbation d'un problème de valeurs propres par modification de la frontière. Cas de la propagation des ondes électromagnétiques dans les guides cylindriques. C. R. Acad. Sci. Paris **217**, 261–263 (1943). [MF 11112]

In connection with the propagation of electromagnetic waves in cylindrical wave guides, one is required to solve the equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2+\alpha^2u=0$, the boundary conditions being $u=0$ and $\partial u/\partial n=0$ for waves of electric and magnetic type, respectively. The author discusses the perturbations of the characteristic values α^2 which result from modifications of the frontier C_0. If h is the perpendicular distance from C_0 to C, the new frontier, φ the perturbation of u, $\varphi=u-u_0$, and ϵ the perturbation of α_0^2, then φ satisfies the equation $\Delta\varphi+\alpha_0^2\varphi+\epsilon u_0=0$, with frontier conditions $\varphi=-h(\partial u_0/\partial n)$, $\partial\varphi/\partial n=-h(\partial^2u/\partial n^2)$ on C_0, the two conditions corresponding to the cases of electric and magnetic waves, respectively. The corresponding values of ϵ are given by

$$\epsilon\iint_{D_0} u_0^2d\sigma+\int_{C_0} h(\partial u_0/\partial n)^2ds=0,$$

$$\epsilon\iint_{D_0} u_0^2d\sigma+\int_{C_0} -hu_0(\partial^2u_0/\partial n^2)ds=0,$$

respectively. A qualitative discussion is made of the case where C_0 is a circular cylinder, and longitudinal grooves are inserted in the boundary. Analogies are cited between these guides and certain anisotropic optical media. *J. W. Green* (Aberdeen, Md.).

***Cartan, Henri. La théorie générale du potentiel dans les espaces homogènes.** Bull. Sci. Math. (2) **66**, 126–132, 136–144 (1942). [MF 10470]

Riesz and Frostman have shown by investigations of certain generalized potentials that many results in potential theory do not depend on the exact nature of the function determining Newtonian, logarithmic, etc., potentials. The author of this paper shows that many of these theorems are related to the properties of locally compact groups. He considers a locally compact group G, a compact subgroup γ and the locally compact homogeneous space $\dot{G}$ consisting

For errata to the review, see the review of another paper by the author, these Rev. 7, 447.

of the class of left cosets of γ by G. The transformations s of G determine a group of transformations $s\dot{G}$ of $\dot{G}$ into itself, this group being isomorphic to G. Next, the notions of measure on G and $\dot{G}$ are developed from the viewpoint of the equivalence of measure functions and positive linear functionals. Considerable use is made of that measure on G which is invariant to the left by G; this is the Haar measure and is denoted by dx. If this measure is transformed to the right by s of G, a proportional measure $dx/\Delta(s)$ is obtained. (If $\Delta(s)\equiv 1$, G is "unimodular," always the case for compact groups.) Corresponding to dx, a measure $d\dot{x}$ exists on $\dot{G}$, invariant under G. Several methods of "composition of measures" are developed, of which the following is a principal one. Let $\dot{\mu}$ and $\dot{\nu}$ be measures on $\dot{G}$, and let x of G correspond to the coset $\dot{x}$ of $\dot{G}$. Then $\int\int f(x\dot{y})d\dot{\mu}(\dot{x})d\dot{\nu}(\dot{y})$ defines a positive linear functional on $\dot{G}$ and hence a measure, denoted by $\dot{\mu}*\dot{\nu}$. If $\dot{\nu}$ is of the form $h(\dot{x})d\dot{x}$, then $\dot{\mu}*\dot{\nu}$ is of the form $g(\dot{x})d\dot{x}$, where $g(\dot{x})=\int h(y^{-1}\dot{x})d\dot{\mu}(\dot{y})$.

Now let $\dot{\rho}=f(\dot{x})d\dot{x}$ be a measure on $\dot{G}$, where f is summable and lower semicontinuous; f is called the base function of the potential theory. The function $f(\dot{x},\dot{y})=\Delta(\dot{x})f(y^{-1}\dot{x})$ is now defined. Then $f(s\dot{x},s\dot{y})=\Delta(s)f(\dot{x},\dot{y})$, and $f(\dot{x},\dot{y})=f(\dot{y},\dot{x})$. (In case $\dot{G}$ is a Euclidean space, G the displacements thereof, and γ the identity, then $\Delta\equiv 1$ and $f(\dot{x},\dot{y})$ is a function of the distance between $\dot{x}$ and $\dot{y}$.) If $\dot{\mu}$ is a measure on $\dot{G}$, consider $\dot{\mu}*\dot{\rho}$, which can be put in the form $U(\dot{x})d\dot{x}/\Delta(\dot{x})$; $U(\dot{x})$ is defined as the potential of $\dot{\mu}$;

$$U(\dot{x})=U(\dot{x},\dot{\mu})=\int f(\dot{x},\dot{y})d\dot{\mu}(\dot{y});$$

$U(\dot{x},\dot{\mu}_1-\dot{\mu}_2)$ is defined as $U(\dot{x},\dot{\mu}_1)-U(\dot{x},\dot{\mu}_2)$, if this exists. The conventional definition of energy is given, and special equations are derived for the energy in case the function $f(\dot{x},\dot{y})$ can be expressed in the form

$$f(\dot{x},\dot{y})=\int f'(\dot{x},\dot{z})f'(\dot{z},\dot{y})d\dot{z}/\Delta(\dot{z}).$$

The following four theorems having analogues in the ordinary potential theories are proved. (1) If the energy of $\dot{\mu}_1-\dot{\mu}_2=0$, then $\dot{\mu}_1\equiv\dot{\mu}_2$. (2) If $U_1=U_2$ almost everywhere, and if $\dot{\mu}_1$, $\dot{\mu}_2$ lie on compact sets, then $\dot{\mu}_1=\dot{\mu}_2$. (3) If f is a real continuous function on a compact set K, f can be uniformly approximated on K by potentials of distributions of the form $[g(\dot{x})-h(\dot{x})]d\dot{x}$, where g, h are positive and continuous. (4) If, in the space of positive mass distributions of finite energy carried by a compact set, one introduces the metric, square root of energy of $\dot{\mu}_1-\dot{\mu}_2$, then the resulting metric space is complete. The hypotheses for this theorem place certain restrictions on μ which do not apply to theorems 1, 2 and 3. *J. W. Green* (Aberdeen, Md.).

Vasilesco, Florin. Sur une notion nouvelle de capacité d'un ensemble. C. R. Acad. Sci. Paris **216**, 191–193 (1943). [MF 10952]

Let μ be a positive distribution, having a potential v which is bounded on $\bar{E}$, the closure of E. The author defines the capacity-v of E as the upper bound of the positive mass on E whose potential will not exceed v. This proves to be identical with the mass of μ swept (balayé) on E [cf. Vasilesco, C. R. Acad. Sci. Paris **215**, 296–297, 317–319 (1942); these Rev. **5**, 146, 133]. The paper is devoted to an account of the proof of existence and uniqueness of the mass distribution on E with mass equal to the capacity-v. *František Wolf* (Berkeley, Calif.).

Rosenblatt, Alfred. On Green's function for plane domains. Revista Ci., Lima **46**, 473–493 (1944). (Spanish) [MF 11502]

The author previously [Revista Ci., Lima **43**, 291–318 (1941); these Rev. **3**, 124] has discussed bounds on the Green's function for bounded domains in Euclidean 3-space. Analogous bounds are now established for bounded domains in the plane. Of the several results obtained, we quote one. Let P, P' be points of a bounded simply-connected plane domain D, let δ, δ' be the respective distances of P, P' from the boundary of D, and let r be the distance PP'. If $f(z)$ is an analytic function mapping D on the interior of the unit circle and if there exist positive numbers α, β satisfying $\alpha\leqq|f'(z)|\leqq\beta$, then the Green's function $G(P,P')$ for D satisfies $G(P,P')\leqq\frac{1}{2}\log\{1+4\beta^2\delta\delta'/\alpha^2r^2\}$.
E. F. Beckenbach (Austin, Tex.).

Pollard, Harry. One-sided boundedness as a condition for the unique solution of certain heat equations. Duke Math. J. **11**, 651–653 (1944). [MF 11568]

The solution of the boundary value problem

$$(1)\qquad \begin{aligned} &u_{xx}=u_t \quad (-\infty<x<\infty,\ 0<t<c),\\ &\lim_{t\to 0} u(x,t)=f(x) \quad (-\infty<x<\infty) \end{aligned}$$

is shown to be unique if the additional condition

$$(2)\qquad u(x,t)\geqq -M>-\infty \quad (-\infty<x<\infty,\ 0<t<c)$$

is imposed on $u(x,t)$. The paper also proves that if $v(x,y)$ is harmonic in the half plane $y>0$ and satisfies conditions (1) and (2), then prescribing the value of the limit of $v(0,y)/y$ for $y\to\infty$ insures uniqueness of $v(x,y)$.
F. G. Dressel (Durham, N. C.).

Theory of Probability

***Tornier, E. Wahrscheinlichkeitsrechnung und allgemeine Integrationstheorie.** J. W. Edwards, Ann Arbor, Michigan, 1944. viii+160 pp. $3.65.

Reprint of the original published in 1936 by Teubner, Leipzig and Berlin.

***Kamke, Erich. Einführung in die Wahrscheinlichkeitstheorie.** J. W. Edwards, Ann Arbor, Michigan, 1944. vii+182 pp. $3.85.

Reprint of the original which was published in 1932 by S. Hirzel at Leipzig.

***Copeland, Arthur H. The teaching of the calculus of probability.** Notre Dame Mathematical Lectures, no. 4, pp. 31–43. University of Notre Dame, Notre Dame, Ind., 1944.

***Menger, Karl. On the relation between calculus of probability and statistics.** Notre Dame Mathematical Lectures, no. 4, pp. 44–53. University of Notre Dame, Notre Dame, Ind., 1944.

Halmos, P. R. The foundations of probability. Amer. Math. Monthly **51**, 493–510 (1944). [MF 11475]

The author's exposition centers about the measure approach to the subject. *J. L. Doob* (Washington, D. C.).

Montel, Paul. Sur les combinaisons avec répétitions limitées. Bull. Sci. Math. (2) **66**, 86–103 (1942). [MF 10467]

Proofs of results announced previously [C. R. Acad. Sci. Paris **214**, 139–141 (1942); these Rev. **4**, 184].

W. Feller (Providence, R. I.).

Gontcharoff, V. Du domaine de l'analyse combinatoire. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 3–48 (1944). (Russian. French summary) [MF 11117]

Proofs of results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 267–269 (1942); **38**, 283–285 (1943); these Rev. **4**, 102; **5**, 124].

W. Feller (Providence, R. I.).

Dunn, Cecil G. Probability method applied to the analysis of recrystallization data. Phys. Rev. (2) **66**, 215–220 (1944). [MF 11310]

Assuming that all orientations are equally likely, the author derives simple approximate expressions for the probability that a random orientation will differ from a prescribed orientation by less than α radians. Several related problems are treated. *W. Feller.*

Cheng, Tseng-Tung. A new probability function and its properties. J. Amer. Statist. Assoc. **39**, 243–245 (1944). [MF 11203]

In computing the distribution function for the sum of n independent variables each of which is restricted to the unit interval, it is necessary to perform an integration over an excedingly awkward region in n-space. The author shows that this situation can be alleviated by integrating instead over a carefully selected system of auxiliary regions and then relating the resulting integrals to the desired integral. The method is applicable when all of the independent variables are subject to the same distribution. The author applies his method to the case in which the variables have unit probability densities. He obtains the moment generating function for this case. *A. H. Copeland.*

Silberstein, Ludwik. Solution of the restricted problem of the random walk. Philos. Mag. (7) **35**, 538–543 (1944). [MF 11344]

The author formulates the problem of a random walk in a two-dimensional net and points out the analogy between the corresponding difference equation and the heat equation. The considerable literature on the problem seems to have escaped the author's attention. *W. Feller.*

Silberstein, Ludwik. The accumulation of chance effects and the Gaussian frequency distribution. Philos. Mag. (7) **35**, 395–404 (1944). [MF 11002]

Simple and perfectly classical considerations on the n-fold convolution of the rectangular frequency function.

W. Feller (Providence, R. I.).

Wald, Abraham. On cumulative sums of random variables. Ann. Math. Statistics **15**, 283–296 (1944). [MF 11324]

Let $z_1, z_2, \cdots$ be independent chance variables with identical distributions, and let $Z_n = z_1 + \cdots + z_n$. Denote by n the smallest integer for which Z_n lies outside a given interval (a, b). Under certain restrictions on z_i the author obtains approximate evaluations of Prob $\{Z_n > b\}$ and of the characteristic function of n. If the values of z_i are integral multiples of a constant d, exact evaluations are given. *D. Blackwell* (Washington, D. C.).

Eggleton, Philip and Kermack, William Ogilvie. A problem in the random distribution of particles. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 103–115 (1944). [MF 11159]

"This paper deals with problems of the following type: . . . N dimensionless particles are arranged at random on a line of length L. We select a number $n \leqq N$ and some length $l \leqq L$. . . . We call a group of n particles contained in a length l an $n:l$. The problem is to find the average number of $n:l$'s in a length L." Actually, the authors solve that problem in a few lines and then pass to analogous problems in two and more dimensions. For the plane case exact expressions are obtained while in the higher dimensional case the method yields at least approximations.

W. Feller (Providence, R. I.).

Bernstein, Serge. Sur les sommes de grandeurs aléatoires liées de classes (A, N) et (B, N). C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 303–307 (1941). [MF 10986]

Let $\{X_k\}$ be a sequence of random variables with $E\{X_k\}=0$, $\sum_{k=1}^n E\{X_k^2\}=B_n$. The sequence is said to belong to the class (N) (to be normal) if it satisfies a Liapounoff condition

$$(B_n)^{-1-\delta}\sum_{k=1}^{n} E\{X_k^{2+2\delta}\}\to 0$$

for some $\delta>0$. Let $a_{k,n}=a_{k,n}(X_1, \cdots, X_{k-1}, X_{k+1}, \cdots, X_n)$ denote the expectation of X_k for given values of the arguments. The sequence $\{X_n\}$ is said to belong to the class (B) if

$$(*)\qquad (B_n)^{-1}E\left\{\left(\sum_{k=1}^{n} a_{k,n}\right)^2\right\}\to 0;$$

this amounts to saying that the dispersions of $\sum_{k=1}^n X_k$ and $\sum_{k=1}^n (X_k - a_{k,n})$ are asymptotically equivalent. The author studies the joint distribution of $u_n=(B_n)^{-1}\sum_{k=1}^n X_k^2$ and $v_n=(B_n)^{-\frac{1}{2}}\sum_{k=1}^n X_k$. It follows, in particular, that all possible limiting distributions of v_n are of the form

$$\int_{-\infty}^{x}\int_{-\infty}^{+\infty}(2\pi z)^{-\frac{1}{2}}e^{-y^2/(2z)}dh(z)dy,$$

where $h(z)$ is a distribution function.

If, in condition $(*)$, the quantities $a_{k,n}$ are replaced by the expectation of X_k for given $X_1, \cdots, X_{k-1}$, the sequence $\{X_k\}$ is said to belong to the class (A). For sequences which belong to both (A) and (N) it is proved that the random variable

$$\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=1}^{n} X_k^2\right)^{-\frac{1}{2}}$$

is, in the limit, normally distributed. *W. Feller.*

Gnedenko, B. V. On the iterated logarithm law for homogeneous random processes with independent increments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 255–256 (1943). [MF 11186]

In important previous results of Khintchine [Bull. Acad. Sci. URSS Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 487–508 (1939); these Rev. **1**, 344] and the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 3–4 (1942); these Rev. **4**, 103] concerning the growth of homogeneous Markov processes there occurred a certain constant (de-

noted by a in the latter review), the probability meaning of which was not quite clear. The present paper gives a clarification of this point. Thus, in certain cases a is simply the variance of the Gaussian component of the process. No proofs are given. *W. Feller* (Providence, R. I.).

Doob, J. L. The elementary Gaussian processes. Ann. Math. Statistics **15**, 229–282 (1944). [MF 11323]

The author considers temporally homogeneous Gaussian Markov (t.h.G.M.) processes, that is, families of chance variables $x(t)$ for which the joint distribution of $x(t_1+h)$, $\cdots$, $x(t_n+h)$ is Gaussian and independent of h, and for which $E(x(t_1), \cdots, x(t_n); x(t_{n+1}))=E(x(t_n); x(t_{n+1}))$ for $t_1<t_2<\cdots<t_{n+1}$. If $x(t)$, $y(t)$ are t.h.G.M. processes, the t.h.G.M. process with variables $(x(t), y(t))$ where every $x(s)$ is independent of every $y(t)$ is called the direct product of the $x(t)$, $y(t)$ processes. A t.h.G.M. process is called deterministic if $E(x(s), x(t))=x(t)$ for $s<t$. It is shown that every t.h.G.M. process is the direct product of deterministic processes of four simple types (three when t is continuous) and a process with no deterministic factors. The structure of the most general t.h.G.M. process without deterministic factors is exhibited for both the discrete and continuous parameter cases. Necessary and sufficient conditions in several forms are obtained for a 1-dimensional t.h.G. process $x_1(t)$ to be a component process of a t.h.G.M. process, that is, for there to exist $n-1$ other 1-dimensional t.h.G. processes $x_2(t), \cdots, x_n(t)$ such that the process with variables $(x_1(t), x_2(t), \cdots, x_n(t))$ is a t.h.G.M. process.

D. Blackwell (Washington, D. C.).

Rice, S. O. Mathematical analysis of random noise. Bell System Tech. J. **23**, 282–332 (1944). [MF 11031]

The theory of "random noise," which is of primary importance in radio engineering, presents many points of mathematical interest inasmuch as it employs both Fourier analysis and theory of probability. This is the first part of a paper dealing with the fundamental aspects of the theory of random noise. It contains mostly results already known and due to a number of investigators. However, the author presents these results in a systematic fashion and with remarkable clarity, thus giving not only a background for his next paper (containing the bulk of original contributions) but also a highly readable account of the origins of the theory.

The mathematical problem is the study of statistical properties of sums

$$(1) \qquad F(t)+\sum_{j=1}^{N} f(t-t_j),$$

where $t_1, t_2, t_3, \cdots$ are independent random variables uniformly distributed in $(0, T)$ and N a random variable independent of the t's and obeying the Poisson law. It is shown that, if $\bar{N}/T$ is large, the statistical properties of (1) are equivalent to those of the "random Fourier series"

$$(2) \qquad T^{-1}\sum_{k=0}^{\infty} B(\omega_k)(X_k \cos 2\pi\omega_k t+Y_k \sin 2\pi\omega_k t),$$

where $\omega_k=K/T$ and $X_0, X_1, Y_1, \cdots$ are independent, normally distributed (with mean 0 and variance 1) random variables. [See also the following review.] Fourier series of type (2) were introduced before in connection with the black body radiation [A. Einstein and L. Hopf, Ann. Physik (4) **33**, 1096–1115 (1910); M. v. Laue, ibid. (4) **47**, 853–878 (1915); A. Einstein, ibid. (4) **47**, 879–885 (1915); M. v. Laue, ibid. (4) **48**, 668–680 (1915)] and Brownian motions of a suspended mirror [G. E. Uhlenbeck and S. Goudsmidt, Phys. Rev. (2) **34**, 145–151 (1929)]. The author next gives an account of N. Wiener's generalized harmonic analysis and gives applications to sums (1) and to the "random telegraph signal." The relation between the so-called "power spectrum" and the "correlation function" (each being a Fourier transform of the other) is stressed as a means of calculating one when the other is known. Although the author occasionally uses a heuristic approach rather than a rigorous one, he is always careful to point it out, thus contributing greatly to the readability of the paper.

M. Kac (Ithaca, N. Y.).

Hurwitz, H., Jr. and Kac, M. Statistical analysis of certain types of random functions. Ann. Math. Statistics **15**, 173–181 (1944). [MF 10824]

The problem of random noises, with which this paper is concerned, has been described in the preceding review. Using characteristic functions, the authors first prove the following theorem. Let, in formula (1), the t_j be independent random variables having the same probability density $p(t)$; let N be an integer-valued random variable which is independent of the t_j; the numbers of those t_j among the first n which fall within two non-overlapping intervals Δ_1 and Δ_2 are mutually independent; then N is distributed according to the Poisson law. The main part of the paper is concerned with a detailed study of the Fourier coefficients of $F(t)$ and their statistical structure. *W. Feller.*

Mathematical Statistics

***Kendall, Maurice G. The Advanced Theory of Statistics. Vol. I.** J. B. Lippincott Co., Philadelphia, 1944. xi +457 pp. $16.00.

The scattered nature of the literature and the existence of excellent treatises in other branches of applied mathematics have made conspicuous the lack until now of a treatise on theoretical statistics. The appearance of the first volume of such a work claims more attention than books of a more limited scope, and its author must be honored for the greater courage required to undertake the huge task. On the other hand, having accepted the responsibility of the potential influence on future research of a treatise, he may expect more severe criticism of his efforts.

The purpose of this first volume may be regarded as largely preparatory to the main business of theoretical statistics; the preface states that the second will deal with the theory of statistical inference. Although the most controversial topics are thus postponed to the second volume, there will inevitably be strong differences of opinion regarding the first also. One's attitude toward the large amount of space devoted to calculations with moments, the small amount given the theory of probability, the manner in which statistical inference is introduced and the presence of certain mathematical inelegancies will be to some extent a matter of taste. Let us consider some of these aspects of the book.

The place of probability theory in the book, "too little and too late," is indicated by the fact that, of the book's 457 pages, pages 164–185 are devoted to the chapter on probability (and likelihood). Postponing the theory causes difficulties in some of the earlier chapters; thus the "deri-

vation" of the binomial and other distributions by the method of "frequency arrays" is vague and confusing. The amount of probability theory introduced is inadequate to the needs of the theoretical statistician; there is no mention of the law of large numbers, nor of stochastic convergence in general. This omission contributes to the logical fuzziness of many of the sections concerned with order relations among random variables and parameters. The author seems reluctant to use probability concepts even after they have been finally introduced. Many important statistical quantities are defined and their properties derived only for finite populations, when the treatment could as easily be general if it were based on probabilities. For instance, for a population partitioned into a contingency table, various measures of association are defined in terms of the number of individuals in the cells, instead of the probabilities associated with the cells. Here and elsewhere the limitation of the theory to finite populations tends to obscure the difference between sample and population, the tendency being enhanced by the frequent use of the same symbol for a population parameter and for its estimate from a sample, and by commonly occurring phrases like "distribution of a parameter," and once even "statistic of a population." Regression theory also, including partial and multiple correlation, is developed for finite populations, whereas the definition of the regression function as an expected value in a conditional probability distribution would have clarified the meaning of linearity of regression, the distinction between the regression function and a linear function fitted by least squares, and other regression concepts.

Calculations of distributions of functions of several random variables are not always preceded by a clear statement of the joint distribution assumed for these random variables. Thus, the notion of statistical independence except for certain linear restrictions is used in the derivations of some distributions but not defined. The intra-class correlation coefficient is defined only for finite populations, then the distribution of the sample intra-class correlation coefficient is derived for the case of "a normal population in which the intra-class coefficient is λ"; this occurs before the definition of the k-variate normal distribution for $k>2$. Again, the assumed underlying probability structure must be quite hazy to permit the statement pertaining to two population parameters (correlation ratio and multiple correlation coefficient) that "from some points of view the two are formally identical." With the language of probability theory available there is little reason for perpetuating such statistical jargon as "If we measure x_1 and x_2 about their respective means," or "the binomial distribution $f=(p+q)^n$," or "proportion" of "all possible samples."

If we imagine an intelligent reader, who was unaware of the work of Neyman, E. S. Pearson and Wald, and had to base his opinion of statistical inference on the exposition in this volume, he would have to judge it unscientific and arbitrary. In 1943 one could and should have given the reader of a book on "advanced theory" a more satisfactory introduction to statistical inference than this: "Although in the remainder of the present volume Bayes' postulate and the principle of maximum likelihood will not often appear explicitly, we shall frequently use a type of argument which is, in the ultimate analysis, based on them. A certain event or series of events is observed; on a hypothesis H the occurrence of these events is found to be highly improbable; and therefore H is rejected in favor of some hypothesis which makes the observations more probable." The statement that "the school of statisticians which rejects Bayes' postulate has substituted for it an apparently different principle based on the use of likelihood" is misleading if not incorrect. The above mentioned writers and others certainly do not at present use as the basis of their theories of statistical inference either likelihood or Bayes' postulate. (However, it is mentioned in a footnote that an account of Neyman's approach will be given in the second volume.)

The mathematical flavor of this book is peculiar. One is pleased by the introduction and use of such tools as the Stieltjes integral and the characteristic function, and at the same time one may confess to an impression of mathematical awkwardness. There is no point to itemizing here the many details building up to this general impression; perhaps it will suffice to characterize one sort. In the same amount of space a larger degree of useful generality could have been imparted to many definitions and theorems. For instance, besides the matter of finite populations already mentioned, there is the following. Let $\{x_i\}$ be random variables with a joint distribution ($i=1, 2, \cdots, n$) and consider the variance of a linear combination y. We find separate formulae for the cases $y=x_1+x_2$ and $y=x_1-x_2$; these are then specialized to the case of statistical independence; this is followed by the formula for the case $y=\sum a_ix_i$ with independent x_i, but the formula for the general case $y=\sum a_ix_i$ without the assumption of independence, whose derivation is as easy, is not given. A further example: in several special cases random variables of the form $\frac{1}{2}\log(n_2u_1/n_1u_2)$, where u_1 and u_2 have independent χ^2-distributions with n_1 and n_2 degrees of freedom, respectively, are shown to have Fisher's z-distribution with n_1 and n_2 degrees of freedom; the general statement is never made.

The following is an outline of the book by chapters. (1) Univariate and bivariate empirical distributions. Theoretical distributions exemplified by analytic formulas. Stieltjes integration. Transformation of probability densities (called "frequency functions"). Statistical independence. (2) Measures of location and dispersion. (3) Moments and cumulants. Definition of moment-generating and characteristic functions. Sheppard corrections to moments. (4) Theorems about characteristic functions. The problem of moments. (5) Standard distributions: binomial, Poisson, Lexis and Poisson generalizations of the binomial, negative binomial, hypergeometric, bivariate binomial, univariate normal. (6) Karl Pearson's system of univariate distributions. Gram-Charlier series of types A and B. (7) Probability. Random variables. Bayes' theorem. Principle of maximum likelihood. Central limit theorem. (8) The practical problem of obtaining random samples. Standard error. (9) Formulas (many only approximate) for standard errors of certain statistics. Order statistics ("mth values"). (10) Methods of deriving "sampling" distributions. (11) R. A. Fisher's theory of k-statistics. (12) The χ^2-distribution as the limiting distribution of Karl Pearson's chi squared criterion under various null hypotheses. (13) This chapter follows closely the one called "Association of Attributes" in the Yule and Kendall text. (14) Correlation and regression in bivariate case. (15) Partial and multiple correlation. (16) Rank correlation.

The above outline is inadequate to suggest the unconventional nature of parts of chapter 7, for example, the author's views on the relation between the theory of probability and the theory of statistical distributions. According to these, an infinite population must be regarded as gener-

ated from some finite population by a limiting process, and the infinite population is not completely characterized by its distribution function, but the limiting process must also be specified. This stipulation enables him to reconcile the principle of maximum likelihood with Bayes' postulate. Praiseworthy features of the book are the collections, after each chapter, of problems for the reader to work, and the large number of numerical examples worked throughout the text to illustrate every step of the theory.

There are several hundred misprints and minor errors, few of which should trouble the reader. One mistake, which has appeared several times in the literature, is serious because it concerns a fundamental and frequently applied result; it consists of omitting the bracketed phrase from the converse of the first limit theorem [§4.12]. "Let $\{\varphi_n\}$ be a sequence of characteristic functions corresponding to the sequence of distribution functions $\{F_n\}$. Then if $\varphi_n(t)$ converges to $\varphi(t)$ [for all real t and] uniformly in some finite t-interval $|t|<a$, then F_n converges to a distribution function F at all continuity points of F, and φ is the characteristic function of F." The author does not give the forms of the theorem usually more convenient in statistical applications, in which "uniformly in some finite t-interval $|t|<a$" is replaced by "$\varphi(t)$ is continuous at $t=0$," or by "$\varphi(t)$ is a characteristic function." We note here also the omission [§4.9, also §4.11, top of p. 101] of the bracketed word in the theorem: "if a sequence of distribution functions $\{F_n\}$ converges to a [continuous] distribution function F, it does so uniformly." Finally, there is a curious justification of the statement [§2.5(b)] that "the mean of a sum is the sum of the means," in which "frequency functions" are added; if the statement refers to composite populations it is false, if to the sum of random variables, the addition of "frequency functions" makes no sense, and besides there is a correct proof in a later section [§3.36]. *H. Scheffé.*

Wald, Abraham. On a statistical generalization of metric spaces. Rep. Math. Colloquium (2) **5–6**, 76–79 (1944). [MF 11409]

An abstract of this paper was published in Proc. Nat. Acad. Sci. U. S. A. **29**, 196–197 (1943) [these Rev. **4**, 220]. Proof is given that the triangle inequality there suggested as an alternative to Karl Menger's furnishes the betweenness properties of ordinary metric space.
J. L. Vanderslice (College Park, Md.).

Wald, Abraham. Note on a lemma. Ann. Math. Statistics **15**, 330–333 (1944). [MF 11331]

Correction to a paper in Ann. Math. Statistics **13**, 434–439 (1942); cf. these Rev. **5**, 129, in particular the last few lines of the review.

Kosambi, D. D. The geometric method in mathematical statistics. Amer. Math. Monthly **51**, 382–389 (1944). [MF 11144]

The author uses n-dimensional geometry to derive the distributions of the mean, the variance, χ^2, t, F and r for samples from a normal distribution. *R. L. Anderson.*

Wilkins, J. Ernest, Jr. A note on skewness and kurtosis. Ann. Math. Statistics **15**, 333–335 (1944). [MF 11332]

A simple proof is given of the inequality $\alpha_4 \geqq \alpha_3^2+1$, where α_3 and α_4 are the third and fourth moments for zero mean and unit variance. Also, an upper bound $(N-2)/(N-1)^{\frac{1}{2}}$ for α_3 is derived, N being the sample size.
A. M. Mood (Princeton, N. J.).

Brown, L. M. Some parameters of sampling distributions simply obtained. Edinburgh Math. Notes no. **34**, 8–11 (1944). [MF 11303]

The author derives the first three moments of the sample mean and the first moment of the sample variance for samples from a finite population. *A. M. Mood.*

Cornfield, Jerome. On samples from finite populations. J. Amer. Statist. Assoc. **39**, 236–239 (1944). [MF 11201]

The problem of finding sampling moments of moments for samples from a finite population of size N is considerably simplified by introducing a variable a_i which is 1 or 0 according as the ith population value is or is not included in the sample ($i=1, \cdots, N$). The moments of the a's are simply

$$E(a_{i_1}^{r_1}a_{i_2}^{r_2}\cdots a_{i_k}^{r_k})=\binom{N-k}{n-k}\Big/\binom{N}{n},$$

where n is the sample size, $r_\alpha>0$, and $i_\alpha\neq i_\beta$ ($\alpha\neq\beta$).
A. M. Mood (Princeton, N. J.).

Daniels, H. E. The relation between measures of correlation in the universe of sample permutations. Biometrika **33**, 129–135 (1944). [MF 10896]

Given two sets $\{x_i\}$ and $\{y_i\}$ of n sample values, assign to each pair (x_i, x_j) a score a_{ij} and to each (y_i, y_j) a b_{ij} with $a_{ij}=-a_{ji}$ and $b_{ij}=-b_{ji}$. A generalized correlation coefficient Γ is defined as

$$\Gamma=\sum a_{ij}b_{ij}/(\sum a_{ij}^2\sum b_{ij}^2)^{\frac{1}{2}}.$$

If we denote the numerator by C, then any permutation of the x's or y's affects only C. Formulas are given for the variances and covariance of any two different permutations. These are used to show that the correlation between Kendall's τ and Spearman's ρ is

$$2(n+1)/(2n(2n+5))^{\frac{1}{2}}.$$

Also, for the product-moment r, it is proved that $R_{rr'}=r_{xx'}r_{yy'}$, where the primes indicate change in scale ($x\rightarrow x'$ and $y\rightarrow y'$). For large n, two Γ's tend to a normal bivariate distribution with a correlation independent of n and variances of the order n^{-1}. *R. L. Anderson.*

Seares, Frederick H. Regression lines and the functional relation. Astrophys. J. **100**, 255–263 (1944). [MF 11416]

Consider a linear function $Y_1=A+BX_1$. The problem is discussed of inferring the values of A and B from observed values Y and X which differ from Y_1 and X_1 by accidental errors of measurement. The general case is considered in which all the observed values have different weights. Least squares procedures are employed. The nature of the regressions of Y upon X, and of X upon Y, is also considered and it is shown how nonlinear regressions will result from non-Gaussian distributions of X and Y, even when the X_1's and Y_1's are linearly related. The regression coefficients are determined by least squares solutions, with weights. *T. E. Sterne* (Aberdeen, Md.).

Cheng, Tseng-Tung. A simplified formula for mean difference. J. Amer. Statist. Assoc. **39**, 240–242 (1944). [MF 11202]

The mean difference m of a set of quantities is the average of the absolute values of their differences. The author obtains a simple formula for computing m by means of finite integration by parts. He also shows that the root mean square difference M (the square root of the average of the squares

of the differences) differs only trivially from the standard deviation σ. He points out that whatever virtue m may possess is also shared by M and σ. *A. H. Copeland.*

Cochran, W. G. Analysis of variance for percentages based on unequal numbers. J. Amer. Statist. Assoc. 38, 287–301 (1943). [MF 9045]

This article presents a method of estimating the proportion of binomial and of extraneous variation for percentages based on unequal numbers. A procedure for determining the best method of weighting these percentages for an analysis of variance with both one and two-way classifications is then formulated. If the observed fraction f_i, based on a total count of n_i, is an estimate of the true fraction p_i, we may write for the total variance of f_i

$$V(f_i)=(p_iq_i/n_i)+\sigma_i^2,$$

where (p_iq_i/n_i) is the binomial and σ_i^2 the extraneous variation. The true weight for f_i would be $W_i=1/(V(f_i))$. The weighting methods advanced are equal weighting (independent of n_i), binomial weighting (proportional to n_i) and partial weighting (binomial for the small n_i and equal for the upper two-thirds of the n_i). For single grouping with r treatments and s samples of each, the lower limit to the efficiency of equal relative to binomial weighting in determining the true treatment variance σ_t^2 is

$$s(r-1)\bar{n}_h\left(N-\sum_{i=1}^{r} N_i^2/N\right)^{-1},$$

where $N_i=\sum_{j=1}^{s}n_{ij}$, $N=\sum_{i=1}^{r}N_i$ and $\bar{n}_h$ is the harmonic mean of the n_{ij}. If the unweighted analysis is used, the ratio of the binomial to the total variance is estimated by

$$\bar{f}(1-\bar{f})/s^2\bar{n}_h,$$

where $\bar{f}=\sum_{i,j}p_{ij}n_{ij}/rs$ and s^2 is the error mean square. If binomial weighting is used, the binomial variance is estimated by $\bar{f}(1-\bar{f})/\bar{n}_h$ and the extraneous by $(s^2-\bar{f}(1-\bar{f}))/\bar{n}$, where $\bar{n}\neq\sum_{i,j}n_{ij}/rs$. Formulas are also given for two-way classification to determine the efficiency of equal weighting and for equal weighting within treatments and within replications. *R. L. Anderson* (Princeton, N. J.).

Holzinger, Karl J. Factoring test scores and implications for the method of averages. Psychometrika 9, 155–167 (1944). [MF 11097]

The methods hitherto in use in carrying out a factor analysis have proceeded by first factoring correlations, arriving first at the matrix of factor coefficients and then calculating the factor scores. In the present paper the author first factors the individual test scores, arriving first at the factor scores and finally calculating the matrix of factor coefficients. Direct factorization of the test score matrix has been suggested before. Guttman [Psychometrika 9, 1–16 (1944); these Rev. 6, 6] outlined an iterative method for principal axes in this case. For the case of orthogonal centroid factors the present method is iterative and equivalent to the usual centroid method and uses the normalized averages of the test scores and of their residuals as successive factors are isolated. For oblique factors, if the tests can be arranged in subgroups each of which measures a single factor, the situation is even simpler to handle since the method obtains the factors all at once. It is pointed out as an implication of the present method that, if the data are of rank greater than one, then a simple average of the test scores incompletely summarizes the information present. *C. C. Craig.*

Lawley, D. N. The factorial analysis of multiple item tests. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 74–82 (1944). [MF 11121]

The present paper extends the method of approach used in a previous paper [Proc. Roy. Soc. Edinburgh. Sect. A. 61, 273–287 (1943); these Rev. 4, 222] to the case of more than one test with the tests measuring different abilities. As before it is assumed that in each test the passing of each item depends upon the ability x which is normally distributed throughout the population of individuals tested with mean and variance constant for the test. The abilities x_i and x_j for different tests may be correlated. Taking scores as the proportions of items passed on each test, the means, variances and covariances of expected scores are found. Next the assumption that the ability (not the score) on each test is a linear function of a number of factors is explored. An interesting result in the case that a battery of tests involves only one general factor is that if the correlation between test scores is not large an ordinary factor analysis would indicate the presence of two factors, one of which would be largely due to the differences of difficulty between tests, a conclusion which had been reached before by a different approach and verified experimentally by Ferguson [Psychometrika 6, 323–329 (1941)]. The remainder of the paper is devoted to devising a means of eliminating this effect of test difficulty from factorial investigations. This is effected by transferring the observed set of variance and covariances to a new set of coefficients and following methods analogous to those used in the earlier paper. A numerical illustration is given.
C. C. Craig (Ann Arbor, Mich.).

Geary, R. C. Minimum range for quasi-normal distributions. Biometrika 33, 100–103 (1943). [MF 11349]

A univariate probability density $f(x)$ is "quasi-normal" if it is continuous, has a single mode at, say, m, and if, for $x<m$, $f'(x)\geqq 0$, and, for $x>m$, $f'(x)<0$. Let

$$\int_a^{a+b} f(x)dx=\alpha<1.$$

For constant α, b is a function of a and has a minimum when $f(a)=f(a+b)$, as is to be expected. Multivariate densities are similarly treated. Application is made to confidence intervals based on a sufficient statistic.
J. Wolfowitz (New York, N. Y.).

Pearson, E. S. and Hartley, H. O. Tables of the probability integral of the Studentized range. Biometrika 33, 89–99 (1943). [MF 11348]

Given a normal population with variance σ^2. A first sample $\{x_i\}$ of n values with $x_{i+1}\geqq x_i$ is used to estimate the range x_n-x_1, and a second sample $\{X_i\}$ of $v+1$ values is used to estimate σ^2 from

$$s^2=v^{-1}\sum_1^{v+1}(X-\bar{X})^2.$$

These estimates are used to compute the Studentized range $q=(x_n-x_1)/s$. If σ is known, the value $w=(x_n-x_1)/\sigma$ is used. Now ${}_vP_n(Q)$ is the chance that q does not exceed Q and is approximated by the equation

$${}_vP_n(Q)=P_n(Q)+(1/v)a_n(Q)+(1/v^2)b_n(Q),$$

where a_n and b_n are differential equations of second and fourth orders, respectively, in dP_n/dW, and $P_n(W)$ is the

probability that w does not exceed W. We see that

$$\lim_{\substack{v\to\infty \\ s^2\to\sigma^2}} {}_vP_n(Q)=P_n(W)|_Q.$$

Values of P_n, a_n and b_n are given for $Q=0.00(0.25)6.50$, and for $n=3(1)20$, and the upper and lower 1% and 5% points of q given for $v=10(1)20$, 24, 30, 40, 60, 120 and for $n=2(1)20$. These were computed by using differences. Examples are given to illustrate the use of the tables in rejecting aberrant sample values, in setting up quality control charts for the range and in problems requiring the control of the ranges (bomb patterns, for example).

R. L. Anderson (Princeton, N. J.).

Court, Louis M. A reciprocity principle for the Neyman-Pearson theory of testing statistical hypotheses. Ann. Math. Statistics 15, 326–327 (1944). [MF 11329]

If a critical region is most powerful for testing the hypothesis H_1 against the alternative H_2, its complement is most powerful for testing H_2 against H_1. *J. Wolfowitz.*

Harshbarger, Boyd. On the analysis of a certain six-by-six four-group lattice design. Ann. Math. Statistics 15, 307–320 (1944). [MF 11326]

The author discusses a modification of an incomplete 6×6 lattice design. He derives for this design formulae for the analysis of variance and for tests of significance and illustrates the procedure by an example. *H. B. Mann.*

Hotelling, Harold. Some improvements in weighing and other experimental techniques. Ann. Math. Statistics 15, 297–306 (1944). [MF 11325]

This paper is one of the first to deal with the design of experiments in the fields of chemistry and physics, and is concerned with an experiment constantly performed in every chemical laboratory. The experiment is simply the weighing of several small objects on a chemical balance or other weighing device. By weighing the objects in combinations rather than separately, accuracy can be much improved. The author finds, for example, that four weights can be determined by four weighings in combinations as accurately as they could be determined by thirty-two weighings of the four objects singly (eight weighings for each object).

The design problem is to find the best set of combinations for given numbers of objects and of weighing operations. The author solves this problem for several of the simpler cases (small numbers of objects and weighings), and he proves in the general case that the minimum standard error attainable with N weighings is $1/\sqrt{N}$ times the standard error of one weighing operation. The sampling problem is discussed but not specifically treated, the standard error of a weighing operation being assumed to be known. *A. M. Mood* (Princeton, N. J.).

Tintner, Gerhard. An application of the variate difference method to multiple regression. Econometrica 12, 97–113 (1944). [MF 10559]

The author deals with the problem of multiple regression when all variables are subject to error. Let $x_1, \cdots, x_p$ be p variates and denote by x_{it} ($i=1, \cdots, p$; $t=1, \cdots, N$) the value of x_i at the time point t. Furthermore, denote the expected value of x_{it} by m_{it} and $x_{it}-m_{it}$ by y_{it}. It is assumed that the expectations $m_{1t}, \cdots, m_{pt}$ satisfy a linear equation $k_0+k_1m_{1t}+\cdots+k_pm_{pt}=0$ with coefficients independent of t. The problem of multiple regression is that of estimating the unknown coefficients in this linear equation on the basis of the observations x_{it}. The author treats this problem under the assumption that all variates y_{it} ($i=1, \cdots, p$; $t=1, \cdots, N$) are independently distributed and the variance σ_i^2 of y_{it} is independent of t. A serious difficulty in estimating the unknown regression coefficients is caused by the fact that σ_i^2 is unknown. The author proposes to overcome this difficulty by estimating σ_i with the help of the variate difference method. The unknown variance σ_i^2 is then replaced by the empirical approximation V_i given by the variate difference method and the problem of estimating the regression coefficients is treated as if the variance of y_i were known to be equal to V_i. The author follows closely the method of weighted regression, as given by T. Koopmans, and determines the regression coefficients by minimizing

$$\sum_t \sum_i (x_{it}-m_{it})^2/V_i.$$

A numerical example is included. *A. Wald.*

Haavelmo, Trygve. The probability approach in econometrics. Econometrica 12, Supplement, 118 pp. (1944). [MF 10820]

The main purpose of this book is to clarify the role of probability schemes in economic theory. The following is a summary of the contents. Chapter 1 discusses the relationship between abstract models and reality. The idea is stressed that no abstract model can have any practical meaning unless a rule is given as to how to identify the abstract variables with some observable phenomena. Such a rule is, of course, always vague to a certain extent. Chapter 2 turns to the question of permanence of economic laws. The notion of autonomy is introduced and examples are given. The autonomy of a relation is the greater the wider the class of variations (defined in some sense) with respect to which it remains invariant. Chapter 3 contains a discussion of probability and random variables and of the way in which the random element is introduced in economic relations. Chapter 4 contains a brief exposition of the Neyman-Pearson theory of testing hypotheses and that of estimation and then discusses the applicability of these tools to economic theories. An economic theory is usually given by a set of stochastic relationships among certain observable variables. These relations will involve some unknown parameters and some auxiliary random variables which are not directly observable but the joint distribution of which is assumed to belong to a given restricted class of distributions. Such a set of stochastic equations implies that the joint distribution of the observable variables must be an element of a certain restricted class of distributions. Thus the problem of testing an economic theory reduces to the statistical problem of testing the hypothesis that the unknown joint distribution of the observable variables is an element of a given class of distributions. Of course two theories are not distinguishable from the point of view of observations if the set of joint distributions of the observable variables consistent with the theory is the same for both theories.

Chapter 5 turns to the problem of estimating the values of the unknown parameters involved in the system of stochastic equations. The following two points are stressed. (1) The estimation of the parameters must be based on the joint distribution of all the observable variables as implied by the complete set of stochastic equations. The procedure of fitting each equation separately, as is often done, may

lead to seriously biased results and inner inconsistencies. (2) The problem of indeterminate parameters, that is, the question whether single valued consistent estimates of the unknown parameters exist can be answered only on the basis of the joint distribution of all the observable variables. The main part of chapter 5 is devoted to this question. Let $p(x_1, \cdots, x_s, \theta_1, \cdots, \theta_k)$ denote the joint distribution of the observable variables $x_1, \cdots, x_s$ implied by the system of stochastic equations, where $\theta=(\theta_1, \cdots, \theta_k)$ is a set of unknown parameters. Suppose that θ^0 is the true parameter point. Clearly, a single valued consistent estimate of θ cannot exist if there exists another parameter point θ^1 such that $p(x_1, \cdots, x_s, \theta_1^0, \cdots, \theta_k^0)=p(x_1, \cdots, x_s, \theta_1^1, \cdots, \theta_k^1)$ identically in $x_1, \cdots, x_s$. Some rules for investigating the existence of such parameter points θ^1 in the neighbourhood of θ^0 are given in terms of the linear dependence of the partial derivatives $\partial p/\partial\theta_i$. In this connection the author extends the Gramian criterion for linear dependence of functions to functions of several variables. Finally the last chapter contains some remarks and suggestions concerning the problem of predicting future values of the observable variables on the basis of past values of these variables.

Econometricians will find the book stimulating reading and helpful in clarifying the nature of the probability approach. Of special interest is the clear recognition of the fact that in estimating unknown coefficients in a simultaneous system of stochastic equations one has to consider the joint distribution of all the observable variables as implied by the whole system and one must not fit each equation separately. The discussion in chapter 5 adds to the clarification of the problem of indeterminate parameters much discussed in econometric literature. *A. Wald.*

Bernardelli, Harro. The stability of the income distribution. Sankhyā 6, 351–362 (1944). [MF 10617]

The main idea of the paper is to describe the changes in the income distribution in a closed society by means of a simple Markov chain (although the author avoids the term). The ergodic properties of Markov chains then serve as an explanation of the stability of the income distribution (Pareto's law). *W. Feller* (Providence, R. I.).

Müller, M. Note sur le produit de plusieurs probabilités d'extinction appliquées à des groupes de valides ou d'invalides. Mitt. Verein. Schweiz. Versich.-Math. 43, 89–97 (1943). [MF 11457]

Zwinggi, Ernst. Über den Vergleich von Verhältniszahlen. Beispiele für die Anwendung neuerer statistischer Verfahren im Gebiete der Versicherung. Mitt. Verein. Schweiz. Versich.-Math. 44, 71–93 (1944). [MF 11458]

Schärf, Henryk. Über einige Variationsprobleme der Versicherungsmathematik. Mitt. Verein. Schweiz. Versich.-Math. 41, 163–196 (1941). [MF 11451]

The paper presents a methodological discussion of various problems in actuarial mathematics which are of the following type. Three sequences of real numbers $\{R_n\}$, $\{E_n\}$ and $\{T_n\}$ (with $E_n\neq 0$) are given with

$$E_tR_t=E_0R_0+\sum_{s=0}^{t-1} E_sT_s.$$

These equations determine R_s if an initial value and $\{E_s\}$ and $\{T_s\}$ are given, and if E_s and T_s are changed to new values $E_s'=E_s+\delta E_s$ and $T_s'=T_s+\delta T_s$ a new series $\{R_s'\}$ is determined. Formulae for the resulting change $\delta R_s=R_s'-R_s$ are derived. In many actuarial problems the R_s depend on the reserves, the E_s represent the basis of calculations, the T_s are determined by the type of insurance considered. The author applies his formulae to the discussion of the signs of the variations δR_s related to Moser's theorem, to the variation of premium reserves, to the determination of compensating dividends, to the theory of accumulation of capitals and to other related problems. *E. Lukacs* (Berea, Ky.).

Kreis, H. Zerfällung einer Gesamtheit in Aktiven- und Invalidengruppen. Mitt. Verein. Schweiz. Versich.-Math. 41, 205–209 (1941). [MF 11453]

Given a mortality table $l_x=Cf(x)$ and a table of actives $\overline{l_x^{aa}}=kf^{aa}(x)$. If a group is decreasing according to the mortality table l_x it may be split into a group of actives and invalids, using l_x^{aa}, if and only if the ratio $f^{aa}(x)/f(x)$ is a positive proper fraction and is decreasing with increasing x. This amounts to the almost trivial statement that the activity table used to split the group must never produce a negative number of invalids. *E. Lukacs* (Berea, Ky.).

TOPOLOGY

***Monteiro, António and Gomes, A. Pereira. Introduction to the Study of the Notion of a Continuous Function.** Publ. Centro Estudos Mat. Fac. Ci. Pôrto, Inst. para a Alta Cultura, Lisboa, no. 8, 152 pp. (1944). (Portuguese)

[The Portuguese title is Introdução ao Estudo da Noção de Função Contínua. The book has also appeared in volume 28 of the Anais de Faculdade de Ciências do Pôrto.] This book examines the topological structure of spaces which admit of the concept of continuous function. The treatment is self-contained. It should serve as a useful introduction for the advanced graduate student to certain types of mathematical abstractions and to methods of mathematical generalization. From this point of view the development of the subject proceeds with great care and the results seem to be very satisfactory. The first four chapters are of an introductory nature exposing the subsequently needed theory of sets, groups and groupoids. The next chapter considers two principal types of topological spaces: the $\mathfrak{B}$ or neighborhood space of Fréchet characterized by a closure operation $\bar{E}$ on the set E for which $O\cdot\bar{O}=O$ (O being the null-set), (I) $E\subset\bar{E}$, (II) $E\subset F$ implies $\bar{E}\subset\bar{F}$; and the space $\mathfrak{F}$ of Sierpiński which in addition to the above axioms satisfies (III) $\bar{\bar{E}}=\bar{E}$. A principal result is that the topology of a space $\mathfrak{B}$ is completely determined by the collection of closed sets in $\mathfrak{B}$ if and only if $\mathfrak{B}$ is an $\mathfrak{F}$.

Continuity is now introduced using the Cauchy neighborhood definition. It is shown that in a $\mathfrak{B}$ space the continuity of a transformation $y=f(x)$ of the space into itself is characterized by $f(\bar{A})\subset\overline{f(A)}$ for all sets A. For an $\mathfrak{F}$ space the condition is $\overline{f(\bar{A})}=\overline{f(A)}$. These relations suggest possible definitions of continuity where neighborhoods are not available but where there is a topological operation of

"closure" even though very weak. As a matter of fact, continuity may be defined over a partially ordered set, for the relation $\subset$ does not need the complexity of background furnished by a Boolean algebra (of subsets of the space). Even on a space without order in which an operator $^-$ is defined, continuity may be introduced by $\overline{f(\bar{x})}=\overline{f(x)}$, where "=" is an equivalence relation. The book closes with a chapter devoted to a problem discussed by Wiener. It is obvious that a topological space determines uniquely its group of homeomorphisms (and its groupoid of continuous endomorphisms). Suppose now that $\mathfrak{W}$ is a space as yet without topology and that a group G of mappings of $\mathfrak{W}$ into itself is given. The problem is essentially: to what extent does G determine a topology in $\mathfrak{W}$? *E. R. Lorch.*

Vaidyanathaswamy, R. On the lattice of open sets of a topological space. Proc. Indian Acad. Sci., Sect. A. 16, 379–386 (1942). [MF 10982]

The author considers the characterization of the topology of a space, in terms of the lattice of its open sets [cf. also H. Wallman, Ann. of Math. (2) **39**, 112–126 (1938)]. In particular, he formulates the separation axioms T_3 and T_4 of Alexandroff-Hopf in terms of the concept of "semisimple ideal," or ideal P such that, given $x\epsilon P$, there exists t, such that $tx=0$ yet $I\epsilon\ t\cup P$. Various other results are given.
G. Birkhoff (Cambridge, Mass.).

Vaidyanathaswamy, R. The localisation theory in set-topology. Proc. Indian Acad. Sci., Sect. A. **20**, 51–61 (1944). [MF 11268]

Let R be a topological T_1-space [Alexandroff-Hopf, Topologie, vol. I, Springer, Berlin, 1935, p. 59]. Let P be a hereditary, additive property of subsets of R, that is, if a set X of R satisfies P, so does any subset of X; and if X and Y satisfy P, then so does $X+Y$. The family of subsets of R having property P is denoted by the same symbol P. For any family P of subsets of R, the corresponding property P of X is taken to be that $X\epsilon P$. Family P is an ideal if and only if property P is hereditary. If X is a set of R, the local function of P, denoted by $P(x)$, consists of the set of points of R each of which has no neighborhood whose intersection with X has property P. If P are the null sets, then $P(X)=\bar{X}$, the closure of X. Letting I denote the finite sets, $I(X)$ is the derived set of X. Set X is said to be discrete if $I(X)=0$, isolated if $X\cdot I(X)=0$, dense-in-itself if $X\subset I(X)$, scattered if it contains no dense-in-itself subset, closed if $X=\bar{X}$, or $I(X)\subset X$. [Cf. Kuratowski, Topologie, Warszawa, 1933, chap. 1.] It is assumed that R is dense-in-itself.

The class of sets having any one of the above properties, except those of being isolated and being dense-in-itself, forms an ideal, with corresponding local function. For example, the local function of the finite sets is $I(X)$, as is likewise that of the discrete sets. Other ideals are: the nondense (nowhere dense) sets; the sets of the first category (each of which is the union of a denumerable family of nondense sets).

The author introduces $X+P(X)$ as a new closure function for X, thus defining a new topology over the elements of R, which he calls P-topology. Properties of the P-topology are studied. For example, any open set of the P-topology is P-dense-in-itself; a set is P-dense-in-itself if $X\subset P(X)$; the P-scattered sets (sets having no P-dense-in-itself subset) are proved to form an ideal. An ideal P is called compact if $P(X)=0$ implies that $X\epsilon P$. (The converse is true for any ideal.) The ideal I of finite sets need not be compact (unless R is compact). The discrete sets form the minimal compact extension of the finite sets. The minimal compact extension of any ideal P is the family of discrete sets of the P-topology. Supercompact ideals are also considered, namely, ideals P such that $X\cdot P(X)=0$ implies that $X\epsilon P$. *A. B. Brown.*

Shanks, M. E. The space of metrics on a compact metrizable space. Amer. J. Math. 66, 461–469 (1944). [MF 10915]

Given a compact metrizable space X, the author considers the set $M_0(X)$ of all quasi-metrics on X, that is, the set of all continuous real valued functions ρ of two variables in X, satisfying the conditions: (1) $\rho(x,y)\geqq 0$; (2) $\rho(x,y)=\rho(y,x)$; (3) $\rho(x,x)=0$; (4) $\rho(x,y)+\rho(y,z)\geqq\rho(x,z)$. With the norm $\|\rho\|=\max\rho(x,y)$, the set $M_0(X)$ constitutes a semi-linear space. If, in addition, ρ satisfies the condition $\rho(x,y)\neq 0$ for $x\neq y$, then ρ is a metric for X. The metrics form a subset $M(X)$ of $M_0(X)$. The author shows the equivalence of the following propositions for any two compact metrizable spaces X and Y: (a) X and Y are homeomorphic; (b) $M_0(X)$ and $M_0(Y)$ are isomorphic; (c) $M_0(X)$ and $M_0(Y)$ are congruent; (d) $M(X)$ and $M(Y)$ are congruent. *S. Eilenberg* (New York, N. Y.).

Milgram, Arthur N. Some metric topological invariants. Rep. Math. Colloquium (2) **5–6**, 25–35 (1944). [MF 11403]

In a previous note [Proc. Nat. Acad. Sci. U. S. A. **29**, 193–195 (1943); these Rev. **4**, 249] the author outlined proofs for the following two theorems. (1) A metric Peano continuum without an equilateral triple is an arc. (2) For each metric simple closed curve C and each positive integer n there exist $n+1$ points $p_0, p_1, \cdots, p_n$ of C for which dist. $(p_i, p_{i+1})=$dist. (p_0, p_n). The present paper gives the details of the proofs. In concluding remarks the author takes exception to a statement made in a note by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **29**, 107–109 (1943); these Rev. **4**, 223] concerning an approach to (1) by means of a convexification process due to G. Beer. The reviewer observes that (1) is valid for any compact metric continuum (that is, local connectivity need not be assumed).
L. M. Blumenthal (Columbia, Mo.).

Begle, Edward G. Regular convergence. Duke Math. J. 11, 441–450 (1944). [MF 11081]

This paper studies the homology aspects of n-regular convergence [see G. T. Whyburn, Fund. Math. **25**, 408–426 (1935); Amer. J. Math. **57**, 902–906 (1935)]. The chief result is in connection with a theorem due to P. A. White [Amer. J. Math. **66**, 69–96 (1944); these Rev. **5**, 149]. He has considered the space L^n of all lc^n subsets of a compactum R, the topology in R being defined in terms of n-regular convergence; he has shown that this space is actually separable and metric. The present author gives a new proof of this theorem and also shows that this space is topologically complete, thus introducing the possibility of arguments involving Baire category. He shows, for example, that, if R is a Euclidean cube of dimension at least $n+2$ and of diameter unity and if M is the subset of L^n consisting of those subsets of R which are lc^{n+1}, then M is of the first Baire category in L^n.

Using the same methods the author also proves some theorems on the homology properties of the limit of a sequence of lc^n sets which converges n-regularly. Among these we mention the following two results, the first of which extends the homology part of a theorem of G. T.

Whyburn, and the second of which was announced by H. E. Vaughan [abstract, Bull. Amer. Math. Soc. **42**, 337–338 (1936)] and which contains G. T. Whyburn's theorem as a special case. (1) Let (P_i) be a sequence of n-dimensional orientable generalized manifolds which converges $(n-1)$-regularly to a set P of dimension n. Then P is an n-dimensional orientable generalized manifold. (2) Let (P_i) be a sequence of closed, orientable 2-dimensional manifolds which converges 1-regularly to a nondegenerate set P. Then P is a closed orientable 2-dimensional manifold, and, for all sufficiently large i, the sets P and P_i are homeomorphic.
D. W. Hall (College Park, Md.).

Fan, Ky. Caractérisation topologique des arcs simples dans les espaces accessibles de M. Fréchet. C. R. Acad. Sci. Paris **212**, 1024–1026 (1941). [MF 10985]

A simple arc in an accessible space is defined to be the homeomorphic image of a closed interval. A necessary and sufficient condition that a set in an accessible space be a simple arc is that it be separable, irreducibly connected between two points and locally connected. The author asserts that this characterization is new even in a metric space since it does not use compactness. A characterization of a topological ray is also given. *D. Montgomery.*

Hall, Dick Wick. On rotation groups of plane continuous curves under pointwise periodic homeomorphisms. Bull. Amer. Math. Soc. **50**, 715–718 (1944). [MF 11282]

The author shows that, if $f(M)=M$ is a periodic homeomorphism on a locally connected continuum M, then any arc axb in M, where a and b are in different orbits under f, can be replaced by an arc $a'x'b'$ in M which intersects no orbit more than once and joins the orbit of a to the orbit of b. Also, if M is in the plane and $f(M)=M$ is an arbitrary point-wise periodic homeomorphism and R is a component of the nonfixed points under f, then (a) every point of the boundary of R in whose vicinity the period function remains bounded is regularly accessible from R and (b) if f has equicontinuous powers, R has property S. *G. T. Whyburn.*

Young, Gail S., Jr. A generalization of Moore's theorem on simple triods. Bull. Amer. Math. Soc. **50**, 714 (1944). [MF 11281]

The author defines a T_n-set, where n is any nonnegative integer, as a continuum which is the sum of an n-cell g and an arc t, such that $g\cdot t$ is a point which is an end point of t and a relatively interior point of g. He then proves that Euclidean n-space does not contain uncountably many mutually exclusive T_{n-1}-sets. This reduces to a theorem of R. L. Moore in the case when n is 2 [see Proc. Nat. Acad. Sci. U. S. A. **14**, 85–88 (1928)]. *D. W. Hall.*

Sorgenfrey, R. H. Concerning triodic continua. Amer. J. Math. **66**, 439–460 (1944). [MF 10914]

This paper is primarily a study of certain internal properties of continua, especially those which pertain to the decomposability of continua in several specified ways. To this end there are presented definitions of eight types of continua which will be referred to as triods [three of these definitions are due to R. L. Moore and one to G. T. Whyburn]. As examples of the results of this section we mention the following. (1) If a locally compact continuous curve M contains a type 1 triod (that is, a continuum T which is the sum of three continua having a point in common and such that no one of them is a subset of the sum of the other two), then M contains a point O and three arcs OA, OB, OC, no two of which have any point in common other than O. (2) If a compact continuous curve contains a type 1 triod then it is itself a type 1 triod. Thus every compact continuous curve is either an arc, a simple closed curve or a type 1 triod.

The remaining sections of the paper investigate the structure of triods of various types with regard to what the author calls their nuclei, and also theorems concerning continua which are characterized, in part, by failing to be triods. One of these theorems is the following generalization of a result due to R. L. Moore [Proc. Nat. Acad. Sci. U. S. A. **20**, 41–45 (1934)]. Let M be a compact nondegenerate unicoherent continuum which cannot be written as the sum of three continua such that the common part of all of them is both a proper nonvacuous subcontinuum of each of them and the common part of each two of them. Then M is irreducible between some two points. *D. W. Hall.*

Eilenberg, Samuel. Continua of finite linear measure. II. Amer. J. Math. **66**, 425–427 (1944). [MF 10912]

In a previous paper [see S. Eilenberg and O. G. Harrold, Jr., Amer. J. Math. **65**, 137–146 (1943); these Rev. **4**, 172] conditions were established for a continuum X in order that there exist a metrization ρ such that the linear measure $L^1(X,\rho)$ is finite. This note shows that every such metric ρ can be expanded to a convex metric without increasing the linear measure. It is also shown that, if X is a continuum, ρ a metric for X such that $L^1(X,\rho)$ is finite, and R the class of metrics L^1-equivalent with ρ, then there is in R a maximal element which is convex and is the only convex element of R. Two metrics ρ_1 and ρ_2 for a continuum X are L-equivalent if $L(A,\rho_1)=L^1(A,\rho_2)$ for every subset A of X.
D. W. Hall (College Park, Md.).

Eilenberg, Samuel. Singular homology theory. Ann. of Math. (2) **45**, 407–447 (1944). [MF 10918]

The continuous or singular homology theory of a space X can be described as follows: the totality of singular simplexes in X forms an infinite complex $S(X)$ which is closure-finite and one considers the discrete homology groups of finite cycles and topologized cohomology groups of infinite cycles of X; they are invariants of X. According to Lefschetz, a singular simplex in X is a pair (s, T), where s is an oriented simplex and T a mapping $s\to X$. If $B:s\to s'$ is a barycentric mapping of s onto another oriented simplex s' of same dimension, then it is agreed that $(s, T)=(s', TB^{-1})$ or $(s, T)=-(s', TB^{-1})$ according as B preserves or reverses orientation. The unsatisfactory element in this definition is that it causes $S(X)$ to have chains of order 2 whereas the groups of chains ought preferably to be free. The present author overcomes this objection and presents a precise and simple treatment of the singular theory based on an enlarged complex $S(X)$: a singular simplex is defined as a mapping $T:s\to X$, where s is a nondegenerate simplex with ordered vertices; if B is the unique barycentric mapping $s\to s'$ which preserves the order of the vertices, it is agreed that $T=TB^{-1}$. A principal result in the theory is that the singular homology groups of a locally finite polyhedron P are the same as the groups of P converted into a complex $k(P)$ by triangulation. To prove this there is introduced as intermediary a complex $K(P)$ whose simplexes are those of $k(P)$, including the degenerate ones, taken with ordered vertices. It is shown that certain natural chain-mappings $\alpha:K(P)\to k(P)$ and $\beta:K(P)\to S(P)$ are "chain-equivalences," that is, they possess inverses in a certain sense. From this it follows that

the homology groups of $K(P)$ are isomorphic to those of $k(P)$ and to those of $S(P)$. The paper concludes with an attractive treatment of products (Whitney) and some applications in the theory of homotopy groups.

P. A. Smith (New York, N. Y.).

Christie, D. E. Net homotopy for compacta. Trans. Amer. Math. Soc. 56, 275–308 (1944). [MF 11251]

The author introduces generalized homotopy groups which are significant for nonarcwise connected spaces and which bear a relation to Čech homology similar to the relation between Hurewicz's homotopy groups and singular homology. Consider the net $\{\Phi_\lambda; \pi_\lambda^\mu\}$ of the nerves Φ_λ of the open coverings of a compactum R, with associated projections π_λ^μ [cf. S. Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloquium Publ., v. 27, New York, 1942, ch. VI and VII]. A collection $\{t_\lambda\}$, t_λ a mapping of a space S into Φ_λ, is called a space-net mapping of S into R if $t_\lambda \simeq \pi_\lambda^\mu t_\mu$ in Φ_λ ($\simeq$means homotopic). Two mappings $\{t_\lambda\}$ and $\{t_\lambda'\}$ are homotopic if $t_\lambda \simeq t_\lambda'$ for every λ. The (weak) homotopy groups $\Pi_n(R)$ consist of the homotopy classes of space-net mappings of an n-sphere into R. Addition is defined by adding the component mappings for every λ; the base points have to be treated appropriately. A second type of groups, the strong groups $\Pi_n^*(R)$, is defined by requiring all mappings and homotopies to be strong, which means roughly that all the homotopies occuring at the different levels ($=\Phi_\lambda$'s) should be consistent. The same groups can be constructed in a second way. Let R be imbedded in a parallelotope P; let $\{U_\lambda\}$ be the system of all neighborhoods of R in P. A collection $\{t_\lambda\}$, where t_λ maps S into U_λ is called a mapping of S towards R if $t_\lambda \simeq t_\mu$ in U_λ if $U_\mu \subset U_\lambda$. Homotopy of two mappings means homotopy of the component mappings t_λ and t_λ' in the corresponding U_λ. Using for S an n-sphere, homotopy groups are then defined. Again there are related strong groups. Equivalence of these groups and the groups defined above is proved using the Kuratowski mappings of R into Φ_λ. The groups are independent of the base point under a certain weak connectedness condition C^{0*}. An analogue to a well-known theorem of Hurewicz is proved: if R is C^{n-1} then $\Pi_n(R)$ and the integral Čech homology groups are isomorphic; C^k means that there is a cofinal set of Φ_λ's which are c^k (their k-sections can be deformed to a point). Various examples, constructed from $\sin 1/x$ and the solenoid, are considered; they show that the new groups have a wider scope than the classical ones. For absolute neighborhood retracts the new groups give nothing new. Other similar concepts are introduced, for instance, that of neighborhood-homotopy of two mappings into R: the mappings are homotopic in every neighborhood of R. Corresponding homotopy groups are defined. Finally it is pointed out that the space-net definition assigns homotopy groups to any net without underlying space.

H. Samelson.

Bockstein, M. Homological invariants of the topological product of two spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 339–342 (1943). [MF 11175]

The author modifies his previous definition of the modular spectrum of a locally compact topological space [C. R. (Doklady) Acad. Sci. URSS (N.S.) 37, 243–245 (1942); these Rev. 5, 48] by requiring some additional homomorphisms. He then shows that the spectra of two spaces A and B determine the spectrum of the Cartesian product $A\times B$. As a corollary it is shown how the homology dimensions of $A\times B$ can be calculated once the dimensions of A and B (with arbitrary coefficient groups) are known.

S. Eilenberg (New York, N. Y.).

Samelson, Hans. Remark on a paper by R. H. Fox. Ann. of Math. (2) 45, 448–449 (1944). [MF 10919]

An example is given to show that a statement by Fox [Ann. of Math. (2) 44, 40–50 (1943); these Rev. 4, 224] is incorrect. A revised proof is given of one of Fox's theorems.

H. M. Gehman (Buffalo, N. Y.).

de Backer, S. M. The four-colour problem. Nature 153, 710 (1944). [MF 10885]

The author outlines a proof by induction of the obvious theorem that any map all of whose regions are triangles can be colored in four colors. He claims "it is not difficult to show that any map . . . can be reduced to a system of connected triangles." In his "complete paper . . . submitted . . . elsewhere" the author may have deduced the coloration in the general case from the result for triangles, but this is not done in the present paper.

P. Franklin (Cambridge, Mass.).

***Reidemeister, K. Topologie der Polyeder und kombinatorische Topologie der Komplexe.** J. W. Edwards, Ann Arbor, Michigan, 1944. ix+196 pp. $5.50.

Reprint of Vol. 17 of the collection Mathematik und ihre Anwendungen in Monographien und Lehrbüchern, published by the Akademische Verlagsgesellschaft in Leipzig. The original appeared in 1938.

GEOMETRY

Busemann, Herbert. Local metric geometry. Trans. Amer. Math. Soc. 56, 200–274 (1944). [MF 11250]

The paper is intended to give a foundation of abstract metric geometry, having as object the treatment of classical problems of differential geometry without differentiability assumptions. It can be regarded as an extension of the author's former work [Metric Methods in Finsler Spaces and in the Foundations of Geometry, Princeton University Press, Princeton, N. J., 1942; these Rev. 4, 109]. A distinct departure from other works and former works of the author is the fact that the metric is not always assumed to be symmetric, a desirable extension owing to the fact that the Finsler space is not symmetric.

The treatment is divided into four chapters. In chapter I the important notion of finite compactness of a space is discussed, which means the property that an infinite bounded set has a limit point. The subsets (not necessarily bounded) and the motions of the space are then metrized and the distances so defined will serve to replace local coordinates and other parameters in analytical arguments. Extremals and geodesics are discussed in chapter II. In order to develop these concepts the space is assumed to be finitely compact

and convex and to have a further property which may be briefly described as the uniqueness of the prolongation in the small. Under these axioms the prolongation is unique even in the large and an extremal is then defined as a certain class of segments. Characteristic properties are given for a continuous curve to "represent" an extremal and it is proved that every extremal admits a representation by a continuous curve. The set of extremals is then topologized. For symmetric spaces it is natural to consider nonoriented extremals, called geodesics, for which many additional properties are established. An abstract theory of conjugate points is developed in chapter III. In chapter IV several applications of the considerations of the former chapters are made, in particular to the problem of characterizing among the so-called G-spaces (which are symmetric) the spaces of constant curvature. Two such characterizations are given, first by the property that the bisectors are linear and the second that a weaker form of the axiom of free mobility is satisfied. As preparation for these considerations a theory of covering spaces is developed, which will also be useful in other problems. *S. Chern.*

Menger, Karl. Projective generalizations of metric geometry. I, II. Rep. Math. Colloquium (2) **5–6**, 60–75 (1944). [MF 11408]

The author introduced the notions of G-oriented and G-metrized groups from which the oriented and nonoriented straight lines arise as special cases [Math. Z. **33**, 396–418 (1931); see also the reviewer's Distance Geometries, Univ. of Missouri Studies, v. 13, no. 2, 1938, pp. 94–97]. The present paper defines and discusses from a similar point of view certain projective spaces, with a view to characterizing projectively subsets of the projective line among all such spaces. A field F is extended by adjoining an ideal element ∞ with $\infty\cdot f=f\cdot\infty=\infty\cdot\infty=\infty$, $0\neq f\epsilon F$; $\infty+f=f+\infty=\infty$, $f\epsilon F$. Denote the extension by F_∞. A set P is called an F-projective space if there is attached to each of its ordered quadruples p, q, r, s (no three identical) an element (p, q, r, s) of F_∞ such that

$$(p, q, r, s)=(q, p, s, r)=(r, s, p, q)$$

and

$$(p, q, r, s)\cdot(p, s, r, q)=(p, q, r, s)+(p, r, q, s)=1.$$

Thus the element (p, q, r, s) has the properties of a cross-ratio. In particular, the set F_∞ becomes an F-projective space, the F-straight $P_1(F)$, by associating with each ordered quadruple their ordinary cross-ratio. Calling a 1:1 cross-ratio-preserving mapping a projectivity, the problem arises of determining necessary and sufficient conditions under which an arbitrary F-projective space is projectively imbeddable in the F-straight $P_1(F)$. This problem is solved in part I. Imbedding theorems for sets of two, three and four elements are followed by a general imbedding theorem which shows that P is projectively imbeddable in P_1 if and only if P contains no quadruple of pairwise distinct elements with a cross-ratio 0 and, for each five elements p, q, r, s, t, $(p, q, r, s)\cdot(p, s, r, t)=(p, q, r, t)$. The reviewer observes that no mention is made of the characterization of the whole F-straight among F-projective spaces.

Part II treats projective spaces with cross-values. A value-set is formed from F_∞ by considering the set $|F_\infty|$ whose elements $|f|$ are unordered pairs $f, -f$ of elements of F_∞ (the element $|\infty|$ of $|F_\infty|$ is taken as the pair ∞, ∞). A set P is an $|F|$-projective space if attached to each ordered quadruple p, q, r, s of P (no three identical) there is an element $|p, q, r, s|$ of $|F_\infty|$, called the cross-value of the ordered quadruple, such that $|p, q, r, s|=|q, p, r, s|=|r, s, p, q|$, $|p, q, r, s|\cdot|p, s, r, q|=|1|$, while $|p, q, r, s|\leq|1|+|p, r, q, s|$ (with $+$, $\cdot$, and $\leq$ suitably defined for values). An important example of such a space is obtained by associating with each ordered quadruple p, q, r, s of F the unordered pair of cross-ratios (p, q, r, s), $-(p, q, r, s)$. Denoting this space by $P_1|F|$ the problem arises of finding necessary and sufficient conditions under which a 1:1 cross-value-preserving mapping exists of an arbitrary $|F|$-projective space on a subset of $P_1|F|$. Such imbedding theorems are obtained. It would be desirable to extend the projective characterizations given for subsets of the projective line to higher dimensions. *L. M. Blumenthal.*

Wylie, C. R., Jr. Hilbert's axioms of plane order. Amer. Math. Monthly **51**, 371–376 (1944). [MF 11142]

As the author says, this is essentially an expository account of the independence of Hilbert's axioms of order, but the approach is fresh and some of the proofs are new. *G. de B. Robinson* (Ottawa, Ont.).

Baer, Reinhold. The fundamental theorems of elementary geometry. An axiomatic analysis. Trans. Amer. Math. Soc. **56**, 94–129 (1944). [MF 10792]

The author refers to the theorems which assert the concurrence of the medians, altitudes, perpendicular bisectors of the sides and the bisectors of the angles of a triangle. The paper is thus a study of the mid-point and orthogonality relations. The mid-point relation causes less difficulty since it only involves the choice of a line at infinity in the projective plane. The author gives a set of five axioms to be satisfied by such a relation and proves that these are categorical. The existence of a mid-point relation is shown to be equivalent to the validity of two properties of the plane, which themselves can be interpreted in terms of the existence and uniqueness of the fourth harmonic point.

An orthogonality relation is equivalent to an involution on the line at infinity. Analytically, $y=xr+s$ is perpendicular to $y=xr'+t$ if and only if $r''=r$. The equivalence of the following theorems is proved: (a) the altitudes of a triangle are concurrent; (b) a theorem concerning circles; (c) F is commutative and associative and $rr'=ss'=c$ for $r\neq0$, $s\neq0$. The constant of orthogonality c is different from zero and depends on the system F of coordinates. Conditions for the equivalence of orthogonality relations and for their uniqueness are given. Finally, a 1-1 correspondence is established between orthogonality relations and congruence relations in the plane. *G. de B. Robinson.*

Landin, Joseph. Axiomatic theory of a singular non-Euclidean plane. I. Rep. Math. Colloquium (2) **5–6**, 53–59 (1944). [MF 11407]

The author shows how the axioms of hyperbolic geometry according to F. P. Jenks [Rep. Math. Colloquium (2) **1**, 45–48 (1939); (2) **2**, 10–14 (1940); (2) **3**, 1–12 (1941); these Rev. **3**, 180] can be modified so as to apply to the case when the absolute degenerates into a line-pair. The result, as the author remarks, is the geometry of the affine half-plane. Each ordinary point lies on one singular line (parallel to the boundary of the half-plane) as well as on a pencil of regular lines. Unless some unexpected feature arises in part II, this geometry does not seem sufficiently interesting to warrant such laborious treatment. *H. S. M. Coxeter* (Toronto, Ont.).

Abbott, James C. The projective theory of non-Euclidean geometry. Rep. Math. Colloquium (2) 5–6, 43–52 (1944). [MF 11406]

[For the earlier parts of this paper, see Rep. Math. Colloquium (2) **3**, 13–27 (1941); (2) **4**, 22–30 (1943); these Rev. **3**, 181; **5**, 152.] Part IV (Theory of Planar Congruency) begins with a complicated definition of reflection, which is found to agree with the usual meaning. A rigid motion (that is, a congruent transformation) is defined as "any one-to-one transformation of the plane into itself preserving points and lines and the relation 'contained in.'" It is proved that any rigid motion is the product of at most three reflections. Hilbert's axioms concerning congruent angles are then verified.

In part V (Embedding Theorem) we find a definition of end-lines (that is, lines at infinity) and of ultra-infinite elements. The difficulty (if it is a difficulty) of defining an ultra-infinite point as a class of ordinary lines is evaded by the unusual device of identifying absolute polar elements, so that an ultra-infinite point is an ordinary line (perpendicular to all the ordinary lines "through" it). With this extended definition of points and lines, the author shows that any two points can be joined by a line, and that any two lines intersect. "Corollary: A non-Euclidean plane can be imbedded in a projective plane." Had this embedding been performed earlier, the treatment of congruence would surely have been facilitated; for example, a reflection could then have been defined as a harmonic homology whose center is the pole of its axis. *H. S. M. Coxeter.*

Weitzenböck, R. Die Kovarianten von vier Ebenen im R_5. Nederl. Akad. Wetensch., Proc. **45**, 139–141 (1942). [MF 10381]

In a previous paper [same Proc. **35**, 1026–1029 (1932)] the author gave the projective invariants of four planes in [5] space. This present work gives their covariants in terms of one point x and these four planes. There are eight such covariants, each a cubic in x, and connected by three quadratic syzygies. Three planes in [5] have no such covariants, and therefore no contravariants, that is, no invariantive forms in three planes and one prime u (of four dimensions). Symbolic methods based on a chain principle are used. *H. W. Turnbull* (St. Andrews).

Weitzenböck, R. Ueber die M_3^3 dreier Ebenen im R_5. Nederl. Akad. Wetensch., Proc. **45**, 215–216 (1942). [MF 10388]

The M_3^3 is a cubic scroll in [5] space, generated by all ∞^2 lines which meet three given planes, or alternatively by the ∞^1 planes (δ) which meet all these lines, where δ is the cross ratio in which each line is cut by the three given planes and by a fourth variable plane, say E_1, E_2, E_3 and E_4. Segre initiated the theory [cf. Mathematische Enzyklopädie III C 7, p. 828]. In 1941, Weitzenböck gave the identity

$$E_4=E_1A_{23}+9\delta J_{\pi 123}-9\delta^2 J_{\pi 213}+\delta^3 E_2 A_{31}=0,$$

connecting the invariants of four planes π, 1, 2, 3 in [5] [cf. Turnbull, Proc. Edinburgh Math. Soc. (2) **7**, 55–72 (1942); these Rev. **4**, 110 for $[2n-1]$ space]. Here E_1 is the mutual moment of two planes 1 and π, and A_{23} of 2 and 3, while $J_{\pi 123}$ is another invariant, linear in all four planes. The discriminant of this binary cubic in δ leads to the equation of the M_3^3 in plane coordinates. Its equations in line and in prime coordinates are also given. *H. W. Turnbull* (St. Andrews).

Weitzenböck, R. Ueber eine Formel aus der Komplexgeometrie. Nederl. Akad. Wetensch., Proc. **45**, 324–326 (1942). [MF 10397]

Add unity to a space of dimension $[n-1]$ and call it G_n. Within this space G_n consider m linear complexes all of whose elements are G_{m-1}'s. Each complex meets a fixed G_m in a null point. What is the condition imposed on G_m for these m null points to be linearly independent, and so form a simplex within G_m? The answer given and proved is that all such G_m's form a complex of order $m-1$. Algebraically a single equation $D^*_{n,m}=0$ expresses this fact, given in complex symbols, and linear in the coefficients of each of the m given complexes. The technique of reduction is determinantal and connected with the Binet-Cauchy theorem and an extension of Bazin's theorem. Examples are given for $m=2$; $m=n$; $m=3$, $n=4$. The condition $D^*_{n,m}=0$ is also given explicitly in terms of the m null points. *H. W. Turnbull* (St. Andrews).

Weitzenböck, R. Ueber gebundene Semiinvarianten. Nederl. Akad. Wetensch., Proc. **45**, 968–969 (1942). [MF 10445]

This is a proof that restricted (gebundene) rational seminvariants of binary forms in a manifold $\mathfrak{M}$ have the same rational basis as the free seminvariants. A seminvariant L is a gradient which is annihilated by the operator

$$\Omega=\sum_{i=0}^{p-1} a_i\partial/\partial a_{i+1},$$

where $a_0, \cdots, a_p$ are the coefficients in the ground form. It is restricted if its a_i satisfy a seminvariant system of equations $S_1=0, \cdots, S_k=0$. Briefly, $\Omega L=0$ in $\mathfrak{M}$. It is free if the coefficient a_1 is absent from it. It is rational if expressible as P/Q, where P, Q are integral seminvariants. The proof depends upon Hilbert's basis lemma, and the use of protomorphs J_i. It renders more precise an earlier work which had proved that the equations $M_i=0$ were equivalent to the equations $J_i=0$ [Weitzenböck, same Proc. **38**, 24–29 (1935)]. *H. W. Turnbull.*

Hadamard, J. A known problem of geometry and its cases of indetermination. Bull. Amer. Math. Soc. **50**, 520–528 (1944). [MF 10844]

In the problem of constructing a square with successive sides on the given points K, L, M, N, the solution is indeterminate if LN and KM are equal in length and mutually perpendicular, that is, if the midpoints of the segments KL, LM, MN, NK are the vertices of a square. A natural generalization of the problem is that of constructing a quadrilateral similar to a given quadrilateral q and having successive sides on K, L, M, N. If q is a parallelogram, indeterminateness in the solution arises when KM and LN intersect at the same angle as the sides of q and are in the same ratio; that is, the midpoints of the segments KL, LM, MN, NK are the vertices of a parallelogram similar to q. The principal analytic device in the treatment is the use of similitudes. The case in which q is a lozenge is treated separately. The general case, q not a parallelogram, is studied. *J. L. Dorroh* (Baton Rouge, La.).

Eves, Howard. Concerning some perspective triangles. Amer. Math. Monthly **51**, 324–331 (1944). [MF 10678]

Let $F_1=0$, $F_2=0$, $F_3=0$ denote equations in two variables. Linear dependencies among the functions $L_1\equiv k_2F_3-k_3F_2$, $L_2\equiv k_3F_1-k_1F_3$, $L_3\equiv k_1F_2-k_2F_1$, $F_1+k_2F_2$, $F_1+k_3F_2$,

$F_2+k_1F_3$, $F_2+k_3F_1$, $F_3+k_2F_1$, $F_3+k_1F_2$ (where the k's denote constants) supply a useful tool for the study of certain geometric properties of figures. The present paper illustrates the application of the method in establishing perspectivities between a given triangle and various associated triangles. One theorem on circles is given.

J. L. Dorroh (Baton Rouge, La.).

Walls, Nancy. An elementary proof of Morley's trisector theorem. Edinburgh Math. Notes no. **34**, 12–13 (1944). [MF 11304]

Wedderburn, J. H. M. On Desargues theorem. Edinburgh Math. Notes no. **34**, 17–19 (1944). [MF 11307]

Brown, L. M. On a chain of circle theorems. Edinburgh Math. Notes no. **34**, 19–20 (1944). [MF 11308]

Bouvaist, Robert et Thébault, Victor. Sur la géométrie du tétraèdre. C. R. Acad. Sci. Paris **217**, 418–419 (1943). [MF 11660]

Convex Domains, Integral Geometry

Jackson, S. B. Vertices for plane curves. Bull. Amer. Math. Soc. **50**, 564–578 (1944). [MF 10849]

The author gives another proof of the four vertex theorem for simple closed curves c with continuous curvature [theorem 4.1] and uses his methods (1) "to characterize geometrically, as far as possible, the curves with just two vertices" [theorem 5.1], (2) to prove that every curve c which meets any circle or straight line at most four times has exactly four vertices [theorem 6.1] and (3) to prove that a curve c which meets a circle $2n$ times and which satisfies an additional condition about the arrangement of the points of intersection has at least $2n$ vertices [theorem 7.1]. Of special importance is the following lemma. "If a Jordan curve bounding a simply connected region is divided in any manner into three arcs, there exists a circle interior to the region and having points in common with all three arcs."

It may be of interest here to refer to the literature which has escaped the author's attention. The arcs of monotone curvature which he discusses in §2 were studied before by Kneser [Heinrich Weber Festschrift, Teubner, Leipzig, 1912, pp. 170–180] and Vogt [J. Reine Angew. Math. **144**, 239–248 (1914)] and from a slightly different point of view by Juel [Danske Vid. Selsk. Skr. (7) **8**, pp. 365–383 (1911)] and Hjelmslev [Oversigt K. Danske Vid. Selsk. Forhandlinger **1914**, no. 1, pp. 3–74]. Vogt's paper also contains two theorems [Satz 9 and Satz 3] which essentially imply the author's lemmas 4.1 and 4.2 [see also below].

Now, in the reviewer's opinion, the four vertex theorem and the problems connected with it will achieve their right perspective only from the viewpoint of the "géométrie finie"; by means of a stereographic projection the above theorems are equivalent to theorems on spherical curves. An ovaloid shall be a bounded closed convex surface which meets every straight line in at most two points. From a paper by Mohrmann [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1917**, 1–4], it follows that a simple closed curve on an ovaloid with continuous tangents and osculating planes and without cusp points has at least four "inflection points." This generalizes theorem 4.1, which the author attributes to Fog and Graustein. By coupling Mohrmann's methods with a generalization of lemma 3.1, an analogous generalization of theorem 7.1 can be obtained. Theorem 6.1 is a corollary of the well-known classification of the closed skew curves of real linear order four [cf., e.g., Linsman, C. R. Acad. Sci. Paris **204**, 463–465 (1937)], if one observes that a curve on an ovaloid cannot have certain types of singularities.

That not even the géométrie finie may furnish the most general approach to the four vertex theorem, and that its content is basically topological has at least been made plausible by Haller's topological generalization of the concept of "arc of monotone curvature" [S.-B. Phys. Med. Sozietät Erlangen **69**, 215–218 (1937); cf. also Hjelmslev's and Haupt's papers quoted there].

P. Scherk.

Santaló, L. A. Note on convex spherical curves. Bull. Amer. Math. Soc. **50**, 528–534 (1944). [MF 10845]

A closed curve on the unit sphere is called convex if it cuts every great circle in at most two points. Such a curve K divides the sphere into two parts, one of which, the "interior of K," lies entirely in a suitable hemisphere. Let $F\leqq 2\pi$ denote its area and let $L\leqq 2\pi$ be the length of K. We denote the arc length by s and the angle between a fixed and a variable great circle of support by τ. Thus a point A on K is determined by the corresponding value of s or of τ. The point A' on the great circle of support of K at A shall have the distance $\pi/2$ from A. The two great circles of support of K through A' form a lune whose angle α is called the breadth of K at A. Then

$$L=\oint \sin \alpha d\tau-\oint \cos \alpha ds.$$

If the great circle through A perpendicular to AA' has a segment of length h in common with the interior of K, then by a duality

$$2\pi-F=\oint \sin h ds+\oint \cos h d\tau.$$

From these formulas a wealth of interesting corollaries is derived for which the reader must be referred to the paper itself.

P. Scherk (Saskatoon, Sask.).

Liberman, J. Geodesic lines on convex surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 310–313 (1941). [MF 10990]

Let S be the boundary of a convex body with interior points in Euclidean n-space. Any two points a, b of S can be connected by a shortest arc C on S. Extending the results of H. Busemann and W. Feller [Acta Math. **66**, 1–47 (1936)] concerning surfaces in 3-space and points of C where S has a tangent plane, it is shown that C has everywhere a right hand and a left hand tangent. If at a point $x\neq a, b$ of C the ray ρ is a right hand tangent and the complementary ray ρ' to ρ lies on the tangent cone to S at x, then ρ' is the left hand tangent of C at x. The tangent exists except at a denumerable number of points and is continuous where it exists. The proofs are based on a new and elegant geometric idea.

H. Busemann.

Hadwiger, H. Über Parallelinvarianten bei Eikörpern. Comment. Math. Helv. **15**, 33–35 (1943).

If G_i, $i=1, 2$, are two convex bodies, and M_i, F_i and V_i are the total curvature integral, area and volume, respectively, of G_i, the author considers functions of M_1, F_1, V_1,

M_2, F_2, V_2 which are invariant under transformations of the G_i into $G_i(\xi)$, the parallel bodies at distance ξ. In view of Steiner's formulas for $M_i(\xi)$, $F_i(\xi)$, $V_i(\xi)$, the five quantities $X=M_2-M_1$, $Y_i=M_i^2-4\pi F_i$, $Z_i=M_i^3-6\pi F_iM_i+24\pi V_i$ are seen to be invariant, and it is pointed out that they form a complete set of such invariants. The so-called decomposition integral

$$\Lambda=\int\{k[G_1,G_2]-1\}\dot{G}_2$$

is now shown to be an invariant. Here $k[G_1,G_2]$ is the number of separate continua in the intersection of the surfaces of G_1 and G_2, where G_1 is held fixed and G_2 moved about; $\dot{G}_2$ stands for the "kinematic density" of G_2, that is, the five dimensional volume element of the three Cartesian and two angle variables fixing the position of G_2. The formula for Λ in terms of X, Y_i, Z_i is not developed except in the special case when G_i is the body parallel to the line of length a_i at distance R_i, that is, a cylinder with hemispherical ends. This formula indicates incidentally that $\Lambda=0$ when $a_1=a_2$ and $R_1=R_2$, answering in the negative a question raised by Bonnesen and Fenchel as to whether two spheres are the only pair of congruent convex bodies all of whose surface intersections are connected curves.

J. W. Green (Aberdeen, Md.).

Dinghas, Alexander. Zum isoperimetrischen Problem für die nichteuklidischen Geometrien. Math. Ann. 118, 636–686 (1943). [MF 10720]

This paper supplements a series of papers by E. Schmidt dealing with the isoperimetric problem in spaces of constant curvature [Math. Z. **44**, 689–788 (1938); **46**, 204–230, 743–794 (1940); **47**, 489–642 (1942); **49**, 1–109 (1943); cf. these Rev. **2**, 12, 262; **5**, 106]. The general isoperimetric problem can sometimes be reduced, by methods analogous to Schwarz' "rounding" process, to the following special problem: determine among all solids of revolution inscribed in a given cylinder and of given surface area the one of greatest volume. This special problem is solved here for the n-dimensional Cayley-Klein spaces with the metric form

$$ds^2=\frac{(1+K\sum_i u_i^2)\sum_i du_i^2-K(\sum_i u_idu_i)^2}{(1+K\sum_i u_i^2)^2},$$

the "cylinders" being given by

$$\frac{u_1^2+u_2^2+\cdots+u_{n-1}^2}{(\tan^2\sqrt{K}\,a)/K}-Ku_n^2\leqq 1.$$

The cases $K>0$, $K=0$, $K<0$, corresponding, respectively, to the elliptic, Euclidean, and hyperbolic geometries, are treated uniformly, except for topological considerations. The same problem has been solved previously by the author for "spherical," Euclidean, and hyperbolic spaces [Math. Z. **47**, 677–737 (1942)]. The present elementary treatment makes no use of Gauss' divergence theorem. The problem is solved by reduction to a similar one on a 2-dimensional Riemannian manifold with a line element of the form $ds^2=du^2+\varphi^2(u)dv^2$, the strip $|u|\leqq a$ taking the place of the cylinder.

The solutions are essentially uniquely determined. They turn out to be either of the "lens-shaped" type, composed of two spherical parts, or of the "sphere-cylinder" type, composed of the basic cylinder and two spherical pieces.

F. John (Aberdeen, Md.).

Algebraic Geometry

Keller, Ott-Heinrich. Eine Bemerkung zu den Plückerschen Formeln. Math. Ann. 118, 626–628 (1943). [MF 10718]

The two basic Plücker formulas for an algebraic curve with nodes and cusps may be taken to be $n=m(m-1)-2\delta'-\kappa$ and $p=\frac{1}{2}(m-1)(m-2)-\delta'$, where m, n and p are the order, class and genus of the curve, respectively, δ' is the number of nodes and κ is the number of cusps. The author shows that these formulas hold for a curve with arbitrary singularities if we define $\delta'=\sum\frac{1}{2}i(i-1)$, $\kappa=\sum' i$, where the sum $\sum$ includes all multiple points, distinct or neighboring, i being the multiplicity, and $\sum'$ includes only the satellite points. The proof is of the standard type which proceeds by induction on the number of neighboring points. This number is reduced by a quadratic transformation with one fundamental point at a nonordinary multiple point, and the expressions $n-m(m-1)+2\delta'+\kappa$ and $p-\frac{1}{2}(m-1)(m-2)+\delta'$ are shown to be unchanged by the transformation. *R. J. Walker* (Aberdeen, Md.).

Rischkov, V. Sur les congruences des courbes planes algébriques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 191–193 (1943). [MF 11064]

This is a collection of some of the author's results on congruences of plane algebraic curves in projective space. Proofs are omitted. The planes of the curves P of the congruence (P) envelope a surface (M) thus setting up a point-curve correspondence $M\to P$ and determining at each point M a certain canonical local reference frame. The equation of each P is written in its local coordinates, the coefficients being functions of asymptotic coordinates on (M). The foci of P are determined by its intersection with a certain unique plane curve S of order one greater than that of P and having a double point at M. The equation of S is written explicitly in terms of the coefficients of P, their derivatives and the coefficients of connection of the local frames. The paper then describes certain relationships between P, S, the local tetrahedron and associated configurations. We mention one of the simpler results: the tangents to S at the double point M are conjugate on (M) when and only when the congruence of the polars of P with respect to M is conjugate to (M). Decomposition of S receives particular discussion. A second section is devoted to results for P of second order. An important class of congruences of polars of M with respect to P is defined through the behavior of certain osculating linear complexes. One example is the congruence of directrices of Wilczynski. The corresponding curves P form a congruence studied by B. Su.

J. L. Vanderslice (College Park, Md.).

Todd, J. A. The geometry of the binary quintic form. Proc. Cambridge Philos. Soc. 40, 1–5 (1944). [MF 10797]

The author interprets the 23 invariants of a general binary quintic form in terms of properties of the rational normal quintic curve in [5] and its projection into [4] from a point not on it. *R. J. Walker* (Aberdeen, Md.).

Abellanas, P. F. The formulae of Schubert for the determination of the fundamental numbers of surfaces of second order. Revista Mat. Hisp.-Amer. (4) 3, 164–175 (1943). (Spanish) [MF 10137]

The coefficients of the equation of a point quadric together with those of the equation of the corresponding plane quadric determine a point in $S_{9,9}$. The system of

points in S_{9}, thus associated with the set of all such pairs of quadrics is a nine-parameter variety. An analysis of this variety is used to type quadrics and to develop Schubert's formulae. *J. L. Dorroh* (Baton Rouge, La.).

Edge, W. L. The identification of Klein's quartic. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 83–91 (1944). [MF 11123]

Some forty years ago a paper appeared, the purpose of which was to correct a statement concerning a certain curve [Coble, Trans. Amer. Math. Soc. **4**, 65–85 (1903)]. The correction of this mistake led to further investigation of covariants connected with this curve, including the present paper [Edge, same Proc. Sect. A. **61**, 140–159 (1941); **61**, 247–259 (1942); these Rev. **3**, 184; **4**, 167]. The purpose of this present note is to show that all the properties mentioned in these and in various other papers by different authors are consequences of one fact. It concerns Klein's plane quartic curve belonging to the simple linear group of order 168. In the representation of the Veronese quartic surface F in [5] the quadric polars of F as to the cubic primal generated by its chords were outpolar as to a quadric arising from unravelment. In the present case, this quadric is not only outpolar but also inpolar as to F. If this condition is satisfied, all the properties of the Klein quartic follow. *V. Snyder* (Ithaca, N. Y.).

Chariar, V. R. On harmonic locus of two given quadrics. Bull. Calcutta Math. Soc. **36**, 41–44 (1944). [MF 11215]

This paper is concerned with those transversal lines which meet two given general quadric surfaces harmonically, and also meet two given algebraic curves of orders m, n, respectively. The locus is a ruled surface of order $4mn$, having the given curves to multiplicity $2n$, $2m$, respectively. An earlier paper in which the present author participated [Chariar, V. R. and Chatterji, N., Bull. Calcutta Math. Soc. **34**, 183–185 (1942); these Rev. **5**, 10] considers the case in which the director curves are both straight lines. As an application of the process, the paper also considers the locus of lines which meet three general quadric surfaces in points of a quadratic involution and also meet the same director curves. The locus is of order $6mn$, having the given curves to multiplicity $3n$, $3m$, respectively.
V. Snyder (Ithaca, N. Y.).

Segre, B. On the quartic surface $x_1^4+x_2^4+x_3^4+x_4^4=0$. Proc. Cambridge Philos. Soc. **40**, 121–145 (1944). [MF 10783]

The special quartic surface F: $x_1^4+x_2^4+x_3^4+x_4^4=0$ is shown to contain 48 lines, and relations between pairs and tetrads of these lines are discussed; F also contains 320 irreducible conics, of two types according as the residual intersections of their planes are reducible or not. The pairs of points of F collinear with two given skew lines of F constitute a birational involution on F. This involution has no fundamental points. Sets S of eight such involutions exist with the property that for any curve C of order n on F there is an involution of S which transforms C into a curve of order greater than n. Hence F contains an infinity of rational curves, and S generates an infinite group of birational transformations of F into itself. *R. J. Walker.*

Morin, Ugo. Sulle varietà algebriche a curve-sezioni di genere tre. Ann. Mat. Pura Appl. (4) **21**, 113–155 (1942). [MF 10508]

Castelnuovo [Atti Accad. Sci. Torino **25**, 695–715 (1890)] and Scorza [Ann. Mat. Pura Appl. (3) **16**, 255–326 (1909); **17**, 281–321 (1910)] obtained the following six birationally distinct types of algebraic surfaces in S_r ($r\geqq 3$) whose curve sections made by sets of S_{r-1} of S_r are of genus 3: (1) a surface of irregularity one; three types, each a rational surface representable on a plane by a linear system of (2) quartics, (3) sextics with seven double basis points, (4) hyperelliptic curves; (5) a quartic surface of S_3; (6) a ruled surface of genus 3 of S_3.

The present paper deals with normal, nonconical, algebraic varieties V_3 of three dimensions in S_r, of order n and whose curve sections by sets of S_{r-1} of S_r are of genus 3. In four chapters, a detailed treatment is given of the V_3 whose surface sections by sets of S_{r-1} of S_r are, respectively, the first four Castelnuovo-Scorza types given above. From these the author obtains seven types of rational V_3 with curve sections of genus 3 and whose representations in S_3 yield all of the birationally distinct types of linear systems of surfaces of order $n>4$, nonconical, and with variable curve sections of genus 3. The author lists two more types of V_3 with curve sections of genus 3, the quartic V_3 of S_4 corresponding to type (5) and the locus of ∞^1 planes of genus 3 corresponding to type (6), but does not study these last two types. *T. R. Hollcroft* (Aurora, N. Y.).

Chabauty, Claude. Sur les points rationnels des variétés algébriques dont l'irrégularité est supérieure à la dimension. C. R. Acad. Sci. Paris **212**, 1022–1024 (1941). [MF 10984]

Among other results the author announces the following theorem on algebraic curves C over an algebraic number field. "If an algebraic curve of genus g and reduced rank r does not possess reducible Abelian integrals, then there exists only a finite number of inequivalent rational systems in an algebraic series on C provided its dimension is not greater than $g-r$." The sketch of the proof involves the p-adic theory of Abelian integrals and the existence of a model free from singularities for Picard varieties.
O. F. G. Schilling (Chicago, Ill.).

Zariski, Oscar. Reduction of the singularities of algebraic three dimensional varieties. Ann. of Math. (2) **45**, 472–542 (1944). [MF 10922]

This paper gives the details of the proof, already announced [Bull. Amer. Math. Soc. **48**, 402–413 (1942); these Rev. **3**, 305], which is based on the methods of the arithmetic theory of algebraic varieties, of the existence of a birational transform of an algebraic variety of three dimensions which has no singular points. The proof follows the lines of the author's simplified proof [Ann. of Math. (2) **43**, 583–593 (1942); these Rev. **4**, 52] for surfaces, and is valid for any ground field of characteristic zero; there are only two points at which it is necessary to assume that the field is of characteristic zero, and these are specially noted.

The theorem of local uniformization [Amer. J. Math. **62**, 187–221 (1940); these Rev. **1**, 102] implies the existence of a finite resolving system of any algebraic function field of dimension r, that is, a finite system of projective models such that any zero-dimensional valuation of the function field has a simple center on at least one of the models. (The proof of this statement is contained in a paper by the author at present in the press; the paper on local uniformization quoted above contains a proof valid when the ground field is the field of complex numbers.) The present paper shows how to replace a finite resolving system for a variety of three dimensions by a single resolving model. This is done by observing that any two members of a

resolving system themselves form a resolving system for a set N of valuations of a function field, and hence it is only necessary to prove that, given a resolving system consisting of two varieties V and V' for a set N of valuations, there exists a resolving system for N consisting of a single model. In general, the join of V and V' is not a resolving model for N, but, in the special case in which, given any valuation v of N, the center of v on one of the surfaces is a simple point which is not a fundamental point of the birational correspondence between V and V', it is true that the join is a resolving model. The main object of the paper is therefore devoted to showing that V and V' can be replaced by varieties for which it can be asserted that the join is a resolving model. This preparation of the resolving system is achieved by means of a series of quadratic and monoidal transformations. This involves a lengthy study of these transformations and of the resolution of the singularities of a surface lying on a threefold by means of them. In the course of this work a number of important results are proved, including the theorem that the singularities of a surface on a threefold which lie at simple points of the threefold can be eliminated by quadratic and monoidal transformations; this generalizes a well-known theorem of B. Levi [Atti Accad. Sci. Torino **33**, 66–86 (1897)].

W. V. D. Hodge (Cambridge, England).

Differential Geometry

Kasner, Edward and DeCicco, John. Scale curves in general cartography. Proc. Nat. Acad. Sci. U.S.A. **30**, 211–215 (1944). [MF 10908]

The authors continue their study of scale curves of maps of a surface Σ on a plane π, treating here the nonconformal case. In this case there are ∞^2 scale curves on π, ∞^1 through each point. Using Cartesian coordinates (x, y) in π, and curvilinear coordinates (x, y) in Σ, so that the map is the identity map in (x, y), the locus of centers of curvature of the scale curves through a fixed point P of π is a general cubic curve with a node at P. In certain cases this cubic is degenerate, and the scale curves have special properties. If the scale curves coincide with the totality of straight lines in π, the Gauss curvature of Σ is a rational function of (x, y) and can be constant only if identically zero.

S. B. Myers (Ann Arbor, Mich.).

Ansermat, A. L'application à la géodésie d'un théorème de Tchebychef. Schweiz. Z. Vermessgswes. Kulturtech. **42**, 83–86, 103–105 (1944). [MF 10626]

Tchebychef stated that, in conformal representation of a region of a surface onto the plane, the "best" mapping is one in which the boundary of the mapped region is a scale curve of the mapping. The author, following Darboux, interprets "best" to mean minimizing the integral over the region of the square of the gradient of the logarithm of the scale function. By simple calculations it is shown that the Tchebychef statement is correct in the special case of the mapping of a piece of sphere onto an elliptic region in the plane.

S. B. Myers (Ann Arbor, Mich.).

MacQueen, M. L. The extremals of two invariant integrals. Bull. Amer. Math. Soc. **50**, 503–508 (1944). [MF 10840]

The author shows that, at each point of an analytic surface, a quadric cone, called the cusp-axis cone, and its dual, called the flex-ray conic, may be defined in terms of the cusp-axes and flex-rays, respectively, of the hypergeodesics which are the extremals of two invariant integrals. These two integrals are similar to the integrals employed by J. E. Wilkins [Duke Math. J. **10**, 173–178 (1943); these Rev. **5**, 14].

T. R. Hollcroft (Aurora, N. Y.).

Ghosh, N. N. A matrix-theory of screws in hyperspace. Bull. Calcutta Math. Soc. **35**, 115–125 (1943). [MF 11008]

This is a continuation of a paper on matrix treatment of rigid body motion in hyperspace [same Bull. **32**, 109–120 (1940); these Rev. **3**, 191]. In that paper different quantities relating to a rigid body in hyperspace, including the system of forces acting on such a body, and the distribution of velocities in it, were represented by $(n+1)\times(n+1)$ matrices of a certain type. This representation is not unique, and the present paper is devoted to the discussion of canonical forms of such representation applicable to the two cases mentioned above. The problem being a generalization of one which in three dimensions leads to the theory of screws, a possible generalization of this theory for hyperspace is considered; here the situation depends on whether n is odd or even; in the second case the analogy with three-dimensional screws can not be pushed as far as in the first. There are disturbing misprints.

G. Y. Rainich.

Kutilin, D. I. Direct method of computing the vector and tensor derivatives in orthogonal systems of curvilinear coordinates. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 431–438 (1943). (Russian. English summary) [MF 11263]

When curvilinear coordinates are used, vectors (and tensors) are given by their components with respect to local bases connected with the coordinate system and changing, in general, from point to point. The components of the (absolute) differential differ then from the partial derivatives of the components of the vector by a term correcting for the change of the basis. In the usual form of tensor analysis this correction term involves the three-index symbols; the present paper is based on the remark that if, in the case of orthogonal coordinates in flat three-space, one uses a basis of unit vectors the coefficients of the correction term may be considered as the components of a vector. The author states that this leads to simplified calculations and gives several examples from mechanics using spherical coordinates.

G. Y. Rainich.

MacDonald, Janet. Conjugate nets in asymptotic parameters. Bull. Amer. Math. Soc. **50**, 697–709 (1944). [MF 11279]

The paper investigates various aspects of the theory of conjugate nets on an analytic nonruled surface in ordinary space, using asymptotic parameters. The principal objects of study are the bundle of quadrics each of which has third order contact with both curves of a conjugate net at a point and various canonical configurations defined by W. M. Davis [Contributions to the Theory of Conjugate Nets, Dissertation, Chicago, 1932].

A. Fialkow.

Choudhury, A. C. The invariants of webs of curves in R_n. Bull. Calcutta Math. Soc. **36**, 62–74 (1944). [MF 11352]

Let

$$\frac{du^1}{a_i^1}=\frac{du^2}{a_2^i}=\cdots=\frac{du^n}{a_i^n},\qquad i=1, 2, \cdots, n,$$

determine a linearly independent set of courses in R_n, that is, let $|a_j^i|\neq 0$. This set of courses will be said to form an n-web. The paper has for its purpose finding functions of a_j^i, which are invariant under the transformation $u^i=u^i(V^1, \cdots, V^n)$, $i=1, 2, \cdots, n$, and under $\bar{a}_j^i=\sigma_j a_j^i$ (not summed on j). The commutator $[\Delta_i, \Delta_j]$ ($\Delta_i=a_i^j\partial/\partial u^j$) may be written in the form $[\Delta_i, \Delta_j]=C_{ij}{}^m\Delta_m$. A partial solution of the problem is obtained by showing that the coefficients $C_{ij}{}^m$ are invariant under the first of the transformations. A complete solution is found by obtaining functions of the coefficients invariant under the second.

V. G. Grove (East Lansing, Mich.).

Rossinski, S. Théorème d'existence d'un couple de congruences non-orthogonal, bilatéralement stratifiable dont la congruence des perpendiculaires communes est isotrope. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 5–9 (1943). [MF 11050]

Rossinski, S. Repère orthogonal lié intrinsèquement avec une congruence arbitraire et les conditions pour la stratification d'un couple de congruences. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 54–56 (1943). [MF 11053]

Rossinski, S. Sur le degré d'arbitraire dans le problème d'existence d'un couple de congruences orthogonal bilatéralement stratifiable dont la congruence des perpendiculaires communes est isotrope. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 101–103 (1943). [MF 11056]

The three papers deal with the same question, namely, the existence of pairs of stratifiable congruences of various types, that is, such congruences that the lines of one are the intersections of the focal planes of the other. In particular, the author is concerned with congruences K_1 and K_2 whose congruence of common perpendiculars K_3 is isotropic. The first conclusion drawn is that, if in any general case a solution exists, it depends on a number of arbitrary constants. He then considers a special case in which the solution depends on 9 arbitrary constants. Since the calculations involved are extremely complicated, the author has not been able to verify his surmise that there may not exist any solution more general than the one for this special case; but, if there does exist one, it can depend on at most 11 arbitrary constants. *M. S. Knebelman.*

Weitzenböck, R. und Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. VIII. Nederl. Akad. Wetensch., Proc. **45**, 17–19 (1942). [MF 10372]

[Parts I–V appeared in the same Proc. **43**, 440–448, 548–556, 668–673, 797–804, 805–814 (1940); cf. these Rev. **2**, 17, 159, 160.] This work generalizes the well-known fact that three skew lines in ordinary space define a quadric surface on which they lie. What is the simplest ruled surface F_2 which contains three lines of [4] in general position? It is a cubic surface $F_2{}^3$, given parametrically by the point

$$\{1+\lambda t, t+\lambda t, \lambda\beta(1-t), \lambda t\gamma(t-1), \lambda t\alpha\}$$

with λ, t parameters, and α, β, γ fixed. There are ∞^3 such surfaces, the given lines being the generators $t=0, 1, \infty$. Fix another point on the surface, and ∞^1 such surfaces contain it. A (2,2) relation $Q(Y, Z)=0$ is given for this surface to contain two given points Y, Z besides the three given lines. This leads to a quadratic line complex, symbolically $(2\,a^2\,\pi^2)(2\,p^2\,\pi^2)=0$, which is the locus of those lines π through which pass the planes meeting all three given lines 2, a, p. Let the given lines be three consecutive lines of the surface F_2, and this complex becomes the differential-comitant $(0\,1\,\pi^2)(0\,2^2\,\pi^2)=0$. The notation indicates successive derivatives with regard to the parameter t, as already explained in previous numbers of this series.

H. W. Turnbull (St. Andrews).

Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. IX. Nederl. Akad. Wetensch., Proc. **45**, 184–188 (1942). [MF 10387]

This continues the theory of the ruled surface F in [4], begun and explained in part II [same Proc. **43**, 548–556 (1940); these Rev. **2**, 159]. From any four general lines in [4] four "attached" lines are found by taking the unique transversal line of every three of these four lines. In the limit, when the given four are consecutive generators of a ruled surface F, the transversals become corresponding and consecutive generators of the attached surface (Heftfläche) F', say, of F. This relation is symmetrical. Corresponding generators of F and F' meet at the "attached point" whose locus is the "attached curve."

The differential invariant $Q=O_{13,23}$ of F is the limiting form of the simplest invariant of five lines of [5], explained in an earlier instalment. If Q vanishes for a general F, then F' is a developable surface, with $R\neq 0$. If $R=0$, the "attached curve" is a straight line. Here R is another differential invariant already introduced.

The cubic contravariant $K=(0^2\,1\,2\,3)1_{u'}2_{u'}3_{u'}$ is considered; $K=0$ is the locus of a prime u which cuts four consecutive generators of F in four coplanar points. Two families of planes are found whose axes pass through the "attached point." Each plane of either set meets four consecutive generators, and one plane of either meets five such generators, the five point plane. The four point plane is the typical member of either family, and it meets a fifth associated line, the bi-line (Beigerade). *H. W. Turnbull.*

Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. X. Nederl. Akad. Wetensch., Proc. **45**, 350–353 (1942). [MF 10401]

This number of the series of papers on the ruled surface F of [4] space returns to the case when the invariant $Q\neq 0$. Two families A and B of invariant planes are considered, one in the tangent prime or [3] space and the other in the "bi-prime" (Beiraum). The former ∞^1 family contains the five point plane [see the preceding review] and also the osculating plane of the "attached curve" at the "attached point" H. The axis of this family is found. The "bi-line" is the fifth associated line g_4 of four consecutive generators of the surface F. It is therefore met by the plane transversal of these four, the four point plane. The five point plane of B meets g in a point K. The line HK is the axis of this second family of planes considered. Basic planes of this family are the five point plane and the plane V common to two consecutive bi-primes. *H. W. Turnbull.*

Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. XI. Nederl. Akad. Wetensch., Proc. **45**, 465–470 (1942). [MF 10411]

This is a continuation of part X [see the preceding review] and initiates a general inquiry into the four, five and six point planes, that is, planes which meet a given generator O_4 of the surface F, and touch F with third, fourth or fifth order contact, respectively. At such a point of F an ∞ of four point planes meet and form a quadric cone, whereas, in general, two five point planes meet at this

point. There are at most four points on the generator through which only one five point plane passes.

It is then proved that two four point planes meet in a line and therefore lie in a prime only if this is the tangent prime or in the bi-prime. They then belong to one or the other of the two families of planes discussed in part X.

H. W. Turnbull (St. Andrews).

Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. XII. Nederl. Akad. Wetensch., Proc. **45**, 540–545 (1942). [MF 10418]

The geometry of the four point plane [see the preceding review] is here worked out systematically by applying the well-known results of the general theory of lines in [4] [H. F. Baker, Principles of Geometry, vol. IV, Cambridge, 1925, pp. 113–130] to the limiting case considered in this series of papers. This naturally introduces Segre's quartic primal W_3^4, namely, the locus of a point in [4] through which passes one repeated plane transversal of four given lines, instead of the usual two planes. This repeated plane cuts the Segre quartic in a repeated conic, which passes through the five points of the plane lying on the four lines and their associated line. In the limiting case, under consideration, the "attached plane" of the ruled surface F becomes a double plane of the quartic W, and the axis of the first family of four point planes [part X; see the second preceding review] is a three fold line on W, while the tangent prime of the ruled surface F meets the quartic in the "attached plane" counted four times. All such properties are interpreted in this and the next paper [part XIII; see the following review]. *H. W. Turnbull.*

Bos, W. J. Zur projektiven Differentialgeometrie der Regelflächen im R_4. XIII. Nederl. Akad. Wetensch., Proc. **45**, 669–674 (1942). [MF 10430]

This continues the account begun in part XII [see the preceding review] of the Segre quartic primal W_3^4 in [4] for the ruled surface F_2, giving two more analytical formulae for the quartic. Formula (320), for the quartic in point coordinates and complex symbols, linear in the coefficients of the four consecutive generators which define the primal, is analogous to that of Saddler [J. London Math. Soc. 16, 167–172 (1941), in particular, p. 171; these Rev. 3, 251] in ordinary symbols and for the unlimited case. Planes through an arbitrary generator O_{i3} of F_2 are considered; those which are in neither the tangent prime nor the bi-prime of F_2, belonging to this generator, cut the Segre quartic W in a conic, together with this generator counted twice. Each four point plane naturally cuts W in a conic K counted twice, which passes through P on the generator O_{i3} and the bi-line g. The tangent at P to the conic is the line where the four point plane meets the tangent prime of the ruled surface.

Among other results are the equation of the quintic primal V_3^5 of the ∞^1 five point planes, and the fact that at most five six point planes exist. *H. W. Turnbull.*

Fialkow, Aaron. Conformal differential geometry of a subspace. Trans. Amer. Math. Soc. **56**, 309–433 (1944). [MF 11252]

The paper treats the conformal differential geometry of a subspace of a Riemann space. The method is that of tensor analysis. Its usefulness rests on the fact that a relative scalar can be constructed which does not vanish if the point of the subspace (of dimension greater than 1) is not umbilical. Using this scalar, one is able to derive from tensors of the Riemann space new tensors which are invariant under conformal transformations.

The author develops a conformal tensor analysis and gives the basic formulas of the problem in question, such as: the Bianchi identities, the Frenet equations, the Gauss-Codazzi equations, etc. Analogous to the Riemannian case, a set of conformally invariant differential forms are constructed, whose coefficients are connected by integrability conditions. For conformally Euclidean spaces these differential forms are proved to characterize the subspace up to certain initial conditions. Various other results are given, analogous to known results in Riemannian geometry.

S. Chern (Princeton, N. J.).

Neville, E. H. The genesis of the Codazzi function. J. London Math. Soc. **19**, 23–27 (1944). [MF 11316]

Let γ be a curve on a surface and let $\xi(s)$ be a unit vector defined at each point of γ, s being the arc length on γ. The vector $d\xi/ds$ is orthogonal to ξ and can be written

$$d\xi/ds = b(s)\zeta + c(s)\mathbf{v},$$

where ζ is the unit normal to the surface and $\mathbf{v}$ is a unit vector tangent to the surface and perpendicular to ξ. The author calls $b(s)$ the bilinear curvature and $c(s)$ the swerve of ξ along γ. It is shown (A) that the bilinear curvature is a function of directions and is a symmetric function of the two directions on which it depends.

If $\Phi(\xi_1, \xi_2, \cdots, \xi_n)$ denotes a scalar function of the directions $\xi_1, \xi_2, \cdots, \xi_n$, which are tangent to the surface along γ, then $d\Phi/ds$ depends on the curvature of the curve as well as its direction. However,

$$d\Phi/ds - c_1 da_1\Phi - \cdots - c_n da_n\Phi,$$

where c_i is the swerve of ξ_i and $da_i\Phi$ is the angular derivative of Φ with respect to ξ_i alone, is independent of the curvature and thus defines a function of $n+1$ directions which is called the Laguerre derivative of Φ. The author proves two more theorems: (B) the Laguerre derivative of the rate of change of any function of position on a surface is a symmetric bilinear function of direction; (C) the Laguerre derivative of the bilinear curvature is a symmetric function of the three directions on which it depends. The last is an interpretation of the Codazzi equations. *G. A. Hedlund.*

Lusternik, L. On categories of some arc families. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 131–132 (1943). [MF 11164]

The author considers the subsets T_n of the functional space R of all rectifiable arcs joining two fixed points on a 2-sphere defined in a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 39, 88–90 (1943); these Rev. 5, 273]. It is proved that

$$(n+1) \geqq \operatorname{cat}_R T_n \geqq \frac{n+3}{2},$$

where $\operatorname{cat}_R T_n$ is the minimum number of closed sets contractible in R into which T_n can be decomposed.

S. Eilenberg (New York, N. Y.).

Lusternik, L. A new proof of the theorem about the three geodesics. C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 3–4 (1943). [MF 11049]

The author sketches a new proof of the theorem which states that, for any class C^3 surface of genus zero, one of the following statements is true: (1) there exist three non-self-intersecting geodesics of different lengths; (2) there exists a one-parametric family of such geodesics of equal length

covering the surface and one such geodesic of different length; (3) all the geodesics are closed non-self-intersecting curves of equal length. The proof is based on the paper reviewed above. *G. A. Hedlund* (Charlottesville, Va.).

Chern, Shiing-shen. A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. Ann. of Math. (2) **45**, 747–752 (1944). [MF 11380]

In order to understand the true nature of the Euler-Poincaré characteristic of a (differentiable) manifold, one has to consider it as a topological invariant of a fibre-space invariantly attached to the manifold, namely, of the space of tangent unit-vectors (or "tangent sphere bundle") to the manifold. It is therefore only natural that an intrinsic proof of the Gauss-Bonnet formula (which expresses the Euler-Poincaré characteristic as the integral of a differential form invariantly attached to the Riemannian structure) should involve the consideration of that fibre-space. This is how the author proceeds here; and his proof, as he states, is merely the simplest example of a general method in the differential-geometric study of fibre-spaces, which is developed in the paper reviewed below.

If R^n is a manifold, the tangent sphere-bundle, or space of tangent unit-vectors, is a manifold M^{2n-1}. A vector-field with isolated singular points on R^n appears in M^{2n-1} as a manifold $\bar{R}^n$ of dimension n, having on R^n the projection R^n, but having in M^{2n-1} a boundary (in the sense of combinatorial topology) which consists of a sum of fibres multiplied with integral coefficients (equal respectively to the indices of the singular points of the given vector-field); the sum of these coefficients is then equal to the Euler-Poincaré characteristic. By integrating on $\bar{R}^n$ a suitable differential form, and applying to this integral the theorem of Stokes, one obtains the generalized Gauss-Bonnet formula for Riemannian manifolds. To construct a differential form which has the required properties, and to prove that it has these properties, is a purely algebraic task (but by no means a trivial one), of which the author acquits himself by a skillful application of the Cartan methods and notations in Riemannian geometry. *A. Weil* (São Paulo).

Chern, Shiing-shen. Integral formulas for the characteristic classes of sphere bundles. Proc. Nat. Acad. Sci. U.S.A. **30**, 269–273 (1944). [MF 11040]

Corresponding to a differentiable manifold there is a "tangent sphere bundle," and this bundle has "characteristic classes" of each dimension, as discovered by E. Stiefel and the reviewer. It is the purpose of the author to express these classes so far as possible in differential geometric terms. Using the method of moving frames of É. Cartan, a condition is given that an exterior differential form "belong to the sphere bundle." The even dimensional characteristic classes are determined in terms of differential forms (not the odd dimensional ones, since the latter involve torsion). It is stated that results on the "normal sphere bundle" of a submanifold of a manifold can be obtained similarly. Proofs are promised later. The study is closely related to Chern's recent proof of the Gauss-Bonnet formula in n dimensions [see the preceding review]. *H. Whitney.*

Rosenson, N. Sur les espaces Riemanniens de classe I. III. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 253–284 (1943). (Russian. French summary) [MF 10941]

In the second part of this paper [same Bull. Sér. Math. **4**, 181–192 (1940); **5**, 325–351 (1941); these Rev. **2**, 163; **3**, 310] the case was studied in which the rank of the second quadratic form $\Omega_{\alpha\beta}$ of the Riemannian manifold is greater than 2. In this paper the case is discussed in which this rank is two. In the case $n\geqq 4$, when a certain given inequality for the Riemann-Christoffel tensor is satisfied, necessary and sufficient conditions are derived that the manifold be of class 1. If these conditions are satisfied, invariant expressions for $\Omega_{\alpha\beta}$ can be found and the manifold cannot be bent. Steps are also given to decide when this inequality is replaced by an equality (fulfilled when $n=3$). For the case of manifolds which can be bent the author refers to the paper by É. Cartan [Bull. Soc. Math. France **44**, 65–99 (1916)]. *D. J. Struik* (Cambridge, Mass.).

Ruse, H. S. On the line-geometry of the Riemann tensor. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 64–73 (1944). [MF 11122]

The family of all lines through the contact point of any tangent space of differentials of a general manifold V_n forms a projective space S_{n-1}, the components of contravariant vectors being homogeneous point coordinates therein. When V_n is Riemannian the metric tensor g_{ij} and Riemann tensor R_{ijkl} define in each S_{n-1} a quadric and a quadratic line complex, respectively. The case $n=4$ gives the most elegant and important results because lines are then self-dual and there are relativistic applications. This case has been studied by previous authors, notably Struik, Lamson and R. V. Churchill. The present author considers briefly the general case and proceeds to special results for $n=3$ and $n=4$. For example, when $n=3$ the quadratic complex in S_2 envelopes a conic (Riemann conic envelope) while the Ricci tensor, if nonsingular, determines a conic directly. This latter is found to be the locus of a point from which the tangents to the Riemann conic envelope and the fundamental conic form an harmonic pencil. Geometric theorems such as this bring forth certain fairly complicated algebraic identities between the components of the Riemann and fundamental tensors. Passing to $n=4$, the lines of the Riemann complex in S_3 passing through a fixed point lie in a surface, the complex cone of the point. Those points for which this cone degenerates form the singular surface of the complex, in general a Kummer quartic surface. The surface is written very simply in terms of R_{ijkl} and dual. The Ricci quadric is now the locus of points whose complex cones are outpolar to the fundamental quadric. Einstein and conformally flat spaces receive simple characterizations through these configurations in S_3. Finally a geometric argument shows the existence in Riemannian V_4 of a remarkable set of 16 null vectors each perpendicular to six of the others. *J. L. Vanderslice* (College Park, Md.).

Wagner, V. The absolute derivative of field of local geometric object in a compound manifold. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 94–97 (1943). [MF 11171]

A compound manifold is an n-dimensional space X_n with associated spaces of dimension m at each point. In the manifold a connection is defined when the associated spaces along a curve of X_n can be represented on each other according to a system of differential equations, whose form generalizes the classical form in that higher derivatives of the coordinates of the points on the curve (with respect to the parameter) are allowed to enter in the equations. A field of local geometric objects is determined by giving a geometric object (of the same kind) in each associated

space. The author then defines the absolute derivatives of different order of such a field and gives, in particular, some results on the group of holonomy of the space.

S. Chern (Princeton, N. J.).

Laptev, G. Sur une classe des géométries intrinsèques induites sur une surface dans un espace à connexion affine. C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 315–317 (1943). [MF 11071]

In an affinely connected space of N dimensions consider a subvariety S_n of n dimensions. An induced connection can be defined on S_n as follows. Take at each point of S_n a set of $N-n$ vectors such that no linear combination of them is tangent to S_n; define the infinitesimal displacement of two neighboring tangent planes of S_n by projection from these $N-n$ vectors. The author gives a characterization of the complex of such induced connections by some abstract properties. It is shown that any affine connection without torsion is equivalent to such an induced connection of a subvariety of n dimensions in an ordinary affine space of $(n^2+3n)/2$ dimensions. *S. Chern* (Princeton, N. J.).

Wade, T. L. Tensor algebra and invariants. I. Nat. Math. Mag. 19, 3–10 (1944). [MF 11150]

This is an expository article in which the author reviews some of the familiar elements of tensor algebra. Tensors of order one and two are introduced and their transformation equations are written in terms of both index and matrix notation. Finally, weighted tensors and the operations of addition, multiplication, contraction and composition are discussed. *N. Coburn* (Austin, Tex.).

Bruck, Richard H. The number of absolute invariants of a tensor. Amer. J. Math. 66, 411–424 (1944). [MF 10911]

The author considers the problem of determining the number of functionally independent absolute invariants of an arbitrary tensor $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ under the general linear group in an n-dimensional space for an underlying field K which is nonmodular. Eisenhart [Continuous Groups of Transformations, Princeton University Press, Princeton, N. J., 1933, pp. 20, 62] has studied this problem for the group of continuous transformations with K the field of real or complex numbers. For this case, Eisenhart showed that the number of such invariants is $N-Q$, where N is the number of independent components of the tensor and Q is the rank of the matrix $\|\xi_{U}{}^{s}\|$, the matrix components being defined by

$$(1)\quad \xi^{(i)\beta}_{(j)\alpha}=[\delta^{i_1}_{\alpha}T^{\beta i_2\cdots i_p}_{(j)}+\cdots+\delta^{i_p}_{\alpha}T^{i_1\cdots\beta}_{(j)}] -[\delta^{\beta}_{j_1}T^{(i)}_{\alpha j_2\cdots j_q}+\cdots+\delta^{\beta}_{j_q}T^{(i)}_{j_1\cdots\alpha}].$$

The author's problem is to find N and Q for the case of the general linear group. In the determination of Q, the following form of Eisenhart's theorem is found to be very useful: the number n^2-Q is the number of independent tensors $F_j{}^i$ which satisfy

$$(2)\quad \xi^{(i)\beta}_{(j)\alpha}F^{\alpha}_{\beta}=0.$$

Thus, it is shown that (2) remains valid when $T^{(i)}_{(j)}$ is replaced by each of its "symmetrized" parts ${}^{[\alpha]}_{[\beta]}T^{(i)}_{(j)}$.

The first part of the paper is concerned with determining N for the various symmetrized tensors. It is shown by use of some previous results [R. H. Bruck and T. L. Wade, Amer. J. Math. **64**, 734–752 (1942); these Rev. **4**, 128] that the number of independent components of such a tensor is $\Omega_\alpha\cdot\Omega_\beta$, where Ω_α is the trace $I^{(i)}_{(i)}$ of the numerical idempotent tensor ${}_{[\alpha]}I$ of symmetry type $[\alpha]$. Furthermore, the number of independent components of an arbitrary bisymmetric tensor is $\binom{n+p-1}{p}$. In connection with the determination of Q, the author shows that, except for some exceptional cases in which $p+q\leqq 2$, if the part ${}^{[\alpha]}_{[\beta]}T^{(i)}_{(j)}$ of the tensor $T^{(i)}_{(j)}$ ranges over all tensors symmetric in the indices (i) and the indices (j) then $Q=n^2$ if $p\neq q$, $Q=n^2-1$ if $p=q$. Some examples are given. Finally, the author extends his theory to include relative tensors. *N. Coburn* (Austin, Tex.).

MATHEMATICAL PHYSICS

★Luneberg, R. K. Mathematical Theory of Optics. Brown University, Advanced Instruction and Research in Mechanics, Providence, R. I., 1944. 491 pp. $4.00.

This publication contains the mimeographed notes of a course on the mathematical theory of optics. The main subject is presented in six chapters. There are two appendices and four supplementary notes.

In chapter I, "Wave Optics and Geometrical Optics," the two theories, usually treated independently, are developed simultaneously. Geometrical optics is shown to be the branch of the general wave theory which describes the propagation of light signals, that is, of sudden discontinuities; in the important case of periodic waves of high frequency, it is represented by approximate solutions of the differential equations of wave optics. Following the introduction of Maxwell's equations, periodic fields are considered, and an integral form of Maxwell's equations is derived. There follow a study of discontinuities and the derivation of Snell's law and Fresnel's formulas. The chapter closes with a discussion of periodic waves of small wave lengths and electromagnetic fields associated with geometrical optics.

Hamilton's theory of geometrical optics is treated in chapter II. After a review of the principles of geometrical optics the canonical equations are derived and Hamilton's characteristic functions V, W and T are discussed. The examples include the mixed characteristic W for stratified media and the angular characteristic T for a spherical mirror and for a refracting spherical surface.

Chapter III presents applications of the theory to special problems like Cartesian ovals, final correction of optical instruments by aspheric surfaces, the angular characteristic for systems of refracting surfaces, media of radial symmetry, Maxwell's fish eye, optical instruments of revolution, spherical aberration and coma, condition for coma free instruments, the condenser problem.

In chapter IV first order optics is developed under the following headings: the first order problem in general, Gaussian optics, orthogonal ray systems, nonorthogonal systems, differential equations of first order optics for systems of rotational symmetry, difference equations for a centered system of refracting surfaces of revolution.

A treatment of the third order aberrations in systems of rotational symmetry is presented in chapter V. After a formulation of the problem, the general types of third order aberrations are derived, their combined effect and

their dependence on the position of the diaphragm discussed. Then the third order coefficients are studied as functions of the positions of the object and the pupil planes, and integral expressions for these coefficients are derived. The chapter closes with a discussion of chromatic aberrations.

Chapter VI, devoted to diffration theory of optical instruments, represents the most advanced and original part of this comprehensive treatise. Based on the results derived in chapter I, a discussion is given of the diffraction of converging spherical waves, of imperfect spherical waves, and of unpolarized light. Diffraction patterns for different types of aberrations are studied, and finally the resolution of two luminous points of equal intensity and the resolution of objects of periodical structure are discussed.

Appendix I contains definitions and theorems of vector analysis. Appendix II gives a method of tracing rays of light in a system of plane reflecting or refracting surfaces.

In the first supplementary note, prepared by N. Chaco and A. A. Blank, the theory of electron optics is developed in close parallelism with the preceding treatment of ordinary optics. The discussion does not include diffraction theory but is confined to the geometrical theory of electron optics under the following subheadings: the equation of movement, the associated Fermat problem, the canonical equations of electron optics, electromagnetic fields of rotational symmetry, first order optics in systems of rotational symmetry, the Gaussian constants of an electron optical instrument, third order electron optics in systems of rotational symmetry, and physical discussion of the third order aberrations of an electron optical instrument.

Supplementary notes II, III and IV are three lectures given by M. Herzberger. The first lecture is on "Optical Qualities of Glass." The second lecture, "Mathematics and Geometrical Optics," discusses the important role of mathematics in geometrical optics and aims at a correct understanding of the work of W. R. Hamilton and its extension. The third lecture, "Symmetry and Asymmetry in Optical Images," points out that the analysis of the complex image of an off-axis point in an axially symmetrical system requires a deeper understanding of image formation than is afforded by a study of the five Seidel aberrations, and offers suggestions on how to gain this insight.

P. Boeder (Southbridge, Mass.).

Herzberger, M. Studies in optics. I. General coordinates for optical systems with central or axial symmetry. Quart. Appl. Math. **2**, 196–204 (1944). [MF 11128]

The purpose of this paper is to find the most general treatment of optical systems with central (point) symmetry and of systems with axial symmetry. The author shows that the various approaches to this problem, as, for example, Hamilton's method of characteristic functions or Herzberger's so-called direct method, can be obtained in a simple manner from the same general principle: Lagrange's fundamental invariants, also known as Lagrangian brackets. A ray is assumed to pass through a number of media separated by refracting surfaces. The points of intersection with the surfaces are determined by vectors a_k, the direction of the ray between two surfaces by vectors $s_{k,k-1}$ of length $n_{k,k-1}$, the refractive index of the medium. From the geometrical relation of these vectors and from Snell's law a simple derivation of Lagrange's invariant is given. The general theory is first applied to systems of central symmetry. It is shown that the general relation between the vectors a, s of the object ray and a', s' of the image ray must have the form

$$a=\alpha l+\beta m,\quad a'=\alpha' l+\beta' m,$$
$$s=\gamma l+\delta m,\quad s'=\gamma' l+\delta' m,$$

where l and m are arbitrary vectors and where $\alpha, \cdots, \delta'$ are scalar functions of one variable and $(l\times m)^2=\tau$. These scalar functions are interrelated by Lagrange's invariant. From these equations the above mentioned different approaches to the problem can be obtained by a special choice of the basic vectors l and m. Letting $l=a$ and $m=s$ the author's direct method is obtained; letting $l=a$ and $m=a'$ leads to Hamilton's point characteristic, etc.

A similar formalism is found if the system has only axial symmetry provided that instead of the vectors a, s; a', s'; l, m their projections onto a plane perpendicular to the axis are introduced in the above formulae. The functions $\alpha, \cdots, \delta'$, however, depend now on three quantities $u=l^2$, $v=m^2$, $w=m\cdot l$. Again it is found that the different methods of treating the problem in question may be obtained simply by specializing the basic vectors l and m. *R. K. Luneberg.*

Nijboer, B. R. A. The diffraction theory of optical aberrations. I. General discussion of the geometrical aberrations. Physica **10**, 679–692 (1943). [MF 11423]

As an introduction to a general diffraction theory of optical instruments [to be published later] the author discusses the geometrical aberrations of an instrument of revolution. The principal aim is to derive a method of classifying these aberrations. The rays which originate at a point y_0, $z_0=0$ of the object plane are characterized, after refraction, by the point of intersection x, y with the plane $z=0$ and by two direction variables ξ, η. The origin of the coordinate system is the Gaussian image point of distance σ from the axis of rotation; the x-axis is determined by a line through the exit pupil point and the Gaussian image point. The coordinates $\xi=r\cos\varphi$, $\eta=r\sin\varphi$ define the point of intersection of a ray with a sphere through the exit pupil point and the Gaussian image point as a center. It is then possible to obtain the intersection x, y with the aid of an aberration function $V(\sigma, \xi, \eta)=V(\sigma, r, \varphi)$ (which is closely related to Hamilton's mixed characteristic) by the relations $x=\partial V/\partial\xi$, $y=\partial V/\partial\eta$. The function V, in systems of revolution, is an analytical function of the three combinations σ^2, r^2, $\sigma r\cos\varphi$ even if the x-axis of the coordinate system does not coincide with the axis of rotation. By developing V an infinite series is obtained with the general term $b_{lmn}\sigma^{2l+m}r^n\cos m\varphi$, in which l, m, n are integers not less than 0, while $n\geqq m$ and $n-m$ is even. Every term defines a certain type of aberration illustrated by the displacement curve

$$x=\tfrac{1}{2}b_{lmn}\sigma^{2l+m}r^{n-1}\times((n+m)\cos(m-1)\varphi+(n-m)\cos(m+1)\varphi),$$
$$y=\tfrac{1}{2}b_{lmn}\sigma^{2l+m}r^{n-1}\times(-(n+m)\sin(m-1)\varphi+(n+m)\sin(m+1)\varphi).$$

This curve may be described by a point moving along a circle the center of which describes a larger circle. On the basis of this Fourier development of the general displacement curves the author obtains a natural method of classifying the geometrical aberrations to any order.

R. K. Luneberg (Buffalo, N. Y.).

Glaser, Walter und Lammel, Ernst. Für welche elektromagnetischen Felder gilt die Newtonsche Abbildungsgleichung? Ann. Physik (5) **40**, 367–384 (1941). [MF 11194]

In ordinary optical instruments of revolution we can assume that object and image space have homogeneous optical properties. This implies, in the realm of first order approximation, the existence of conjugate planes, whose position on the z-axis, that is, on the axis of rotation, is determined by the lens equation. If the two focal points of the instrument are chosen as reference points this equation assumes Newton's form: $zz_1=-nn_1f^2$, where f is a constant of the instrument. For an arbitrary choice of reference points we have the more general projective relation: $z_1=(az+b)/(cz+d)$, in which a, b, c, d are constants. In the case that object and image space are not homogeneous one can still prove the existence of conjugate planes; however, their location is no longer regulated by a simple equation. The electron microscope represents an example since the object lies in a region of great field strength. In view of this the authors investigate the question whether there exist special electromagnetic fields in which the position of conjugate planes is determined by a relation of the above projective character.

In the first part of the paper a relatively simple derivation of the differential equations of first order electron optics for the path $x(z)$, $y(z)$ of a paraxial electron is given. It is then found that the condition for conjugate planes may be written in the form $f(z_1)=f(z)$, where $f(z)=\rho/\sigma$ and $\rho(z)$, $\sigma(z)$ are two fundamental solutions of the linear differential equation of second order for $x(z)$ or $y(z)$. If the relation between z and z_1 is projective then the function $f(z)$ must satisfy the condition $f((az+b)/(cz+d))=f(z)$. The general solution of this functional equation is given. Finally the electromagnetic fields are determined which lead to differential equations for which the quotient ρ/σ is a function of the established type. *R. K. Luneberg.*

Schriever, O. Angleichung der elektromagnetischen Reflexions- und Brechungstheorie an die physikalischen Vorgänge. Ann. Physik (5) **40**, 448–462 (1941). [MF 11192]

The author derives a set of formulae for the reflection and refraction of electromagnetic waves on a plane surface. In the case of metallic media these formulae differ from former results, especially from the formulae of Frenel's type in the case of plane waves. The reason is that the author bases his theory on assumptions which are different from those ordinarily made in metal optics. His assumptions are the following. The reflected and refracted waves can be considered as originating in virtual sources. The light rays from these sources to a point on the boundary have the same optical length as the light ray from the original source to this point. The optical length is thereby determined by a "geometrical index of refraction" n, which is the real part of the complex index of refraction $m=n-in\kappa$. From this latter assumption the author obtains the laws of reflection and refraction. His law of refraction $n\sin\varphi=n'\sin\varphi'$ differs, however, from the usual complex form $m\sin\varphi=m'\sin\varphi'$ of the classical theory. Consequently his angles φ and φ' are real and determine the direction of the refracted waves, whereas the angles of the classical theory are complex and define a more complicated type of wave. The expressions for the amplitudes are finally obtained by assuming the continuity of the tangential components of the electromagnetic vectors on the boundary. *R. K. Luneberg* (Buffalo, N. Y.).

Sivukhin, D. V. Phenomenological theory of the transition layer. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 361–375 (1943). (Russian) [MF 11225]

The two transparent homogeneous isotropic media I and II with refractive indices n_1 and n_2 occupying the half spaces $z\geqq 0$ and $z\leqq -l$, respectively, are separated by a stratified nonisotropic medium of axial symmetry with the principal refractive indices $n_x(z)=n_y(z)$ and $n_z(z)$ occupying $0\geqq z\geqq -l$. A plane light wave of wave length $\lambda=2\pi/k$ and of arbitrary polarization and inclination falls on $z=0$ and is partly reflected and partly transmitted into medium II. This problem is treated on the basis of Maxwell's equations with the usual boundary conditions. The field in the interstitial layer is expanded in terms of k. Manageable expressions are obtained by retaining the terms up to $O((kl)^2)$ inclusive. The result, which is correct for $(l/\lambda)^4\ll 1$, involves simple and iterated zero and first order moments of n_x^2 and n_z^2. Curves are plotted illustrating the conditions for the special case of a power law for the refractive indices in the interstitial layer. The polarization of the reflected ray for incidence close to the Brewster angle is investigated. *H. G. Baerwald* (Cleveland, Ohio).

Leontovič, M. On a method of solving the problem of propagation of electromagnetic waves near the surface of the earth. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 16–22 (1944). (Russian) [MF 11126]

The gist of the method, on which the success as to tractability of more involved propagation problems is based, is the introduction of approximate boundary conditions right at the outset; in this respect, it resembles the approach of Strutt [1929–30], but is more powerful, and it corresponds partly with those of Weyl and van der Pol. The boundary conditions for $z=0$ in question: $E_y=\epsilon^{-\frac{1}{2}}H_x$, $E_x=-\epsilon^{-\frac{1}{2}}H_y$ or, for the Hertzian vector $\Pi_z=\Pi$: $\partial\Pi/\partial z=(ik/\sqrt{\epsilon})\Pi$ are predicated on the assumption of a large modulus of the complex relative dielectric constant ϵ of the soil [or more accurately, $k^2d^2\ll 1$, where d is the penetration (skin effect) constant; G. Grünberg, Phys. Rev. (2) **63**, 185–189 (1943); these Rev. **4**, 205]. Writing, accordingly, the spatial attenuation function $U=Re^{ikR}\Pi$ in terms of the numerical distance $\rho=kr/(2|\epsilon|)$ and height $\zeta=kz/(2|\epsilon|)^{\frac{1}{2}}$, and neglecting the quantity $\beta^2\equiv 1/(2|\epsilon|)$ (β, representing the ratio of horizontal and vertical scales, is associated with the Brewster angle of the direct ray), one reduces the differential equation for U to that for one-dimensional heat flow with a homogeneous mixed boundary condition. The solution of the propagation problem is thus obtained at once and proves identical with that of Weyl and van der Pol, as is to be expected from the correspondence of procedure. However, the method makes it possible to tackle more involved problems such as those of the spherical earth and of "slowly" varying soil constants. Only the latter avenue is pursued; following a discussion of the various permissible neglections that are compatible with $\beta^2\ll 1$, an integral equation of the Volterra type is obtainable for a function closely associated with U. No solution for specific cases of change of ϵ with distance is given. It should be emphasized that due to the assumptions involved the method is of importance for the long and medium wave range only. *H. G. Baerwald.*

Vladimirskii, V. V. The propagation of electromagnetic waves along isolated wires. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 139–149 (1944). (Russian) [MF 11443]

Nikolsky, K. V. On a new theory of electromagnetic field. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 309–312 (1943). [MF 11179]

Bopp [Ann. Physik (5) **38**, 345–384 (1940); these Rev. **2**, 336] and Podolsky [Phys. Rev. (2) **62**, 68–71 (1942); these Rev. **4**, 31] have proposed a modification of the Lagrangian function of classical electromagnetic theory which makes it possible to split the resulting field into two parts, representing the Maxwell field and the Yukawa field, respectively. The author discusses this formulation and shows that it may be used to determine the Dirac quantum brackets [see Podolsky and Kikuchi, Phys. Rev. (2) **65**, 228–235 (1944); these Rev. **5**, 277], and also the spin of the neutral particles in the Yukawa field. *M. C. Gray.*

Fock, V. Electric field in a hollow in a conducting plane. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 249–269 (1943). (Russian) [MF 11218]

The determination of the influence of a conducting cavity of linear dimensions $\ll\lambda$ in the conducting plane $z=0$ reflecting plane electromagnetic waves requires the determination of a potential $\Phi(x, y, z)$ such that $\Phi\sim z$ for $z\rightarrow+\infty$, while $\Phi=0$ at $z=0$ for $r=-\sin\gamma$ and at the adjoining spherical cavity surface of radius 1: $r^2+(z+\cos\gamma)^2=1$ for $z\leqq 0$, γ being the wedge angle, $\pi<\gamma<2\pi$. The simplest case ($\gamma=3\pi/2$: hemispherical cavity) is treated first. Inversion on $(1, 0, 0)$ transforms the boundary into the plane wedge: $\eta=0$, $\xi>0$; $\xi=0$, $\eta<0$ with angle $3\pi/2$. The solution is obtained for a function linearly related to the potential function in terms of polar coordinates ρ, φ, ζ by double application of the real Fourier integral and subsequent use of complex integration which makes it possible to carry out one integration and to make the remaining integral more manageable. The result is then written in toroidal coordinates in which it assumes the simplest and final form; these could be introduced at the outset inasmuch as they are germane to the boundary shape and permit separation of the Laplace equation. The result so far presented as an integral is then reduced to elliptic integrals of the third kind and also expanded in a Fourier series in terms of the toroidal azimuth, involving Legendre functions of either kinds and of orders $\frac{1}{2}$ and its multiples as coefficients. The case of a general wedge angle γ is subsequently treated similarly using the toroidal coordinates ϑ, φ:

$$r=\frac{\operatorname{sh}\vartheta}{\operatorname{ch}\vartheta-\cos\varphi},\quad z=\frac{\sin\varphi}{\operatorname{ch}\vartheta-\cos\varphi}.$$

The result is written: $\Phi=-(2(\operatorname{ch}\vartheta-\cos\varphi))^{\frac{1}{2}}\partial\chi/\partial\varphi$, where the function χ satisfying the equation

$$\frac{\partial^2\chi}{\partial\vartheta^2}+\operatorname{cth}\vartheta\frac{\partial\chi}{\partial\vartheta}+\frac{\partial^2\chi}{\partial\varphi^2}+\frac{\chi}{4}=0$$

and the boundary conditions $\partial\chi/\partial\varphi=0$ for $\varphi=0$ and $\varphi=\gamma$ is obtained in closed form as an integral:

$$\chi=\frac{2}{\gamma}\int_0^\infty\frac{\operatorname{sh}(\pi t/\gamma)}{\operatorname{ch}(\pi t/\gamma)-\cos(\pi\varphi/\gamma)}\frac{dt}{(2(\operatorname{ch}t-\operatorname{ch}\vartheta))^{\frac{1}{2}}}.$$

H. G. Baerwald (Cleveland, Ohio).

Kramers, H. A. On multipole radiation. Physica **10**, 261–272 (1943). [MF 10733]

A new theory of multipole radiation, both classical and quantum theoretical, is developed. The invariance under the complete group of rotations of the radiation field satisfying Maxwell's equations in free space is taken as the point of departure. The electric and the magnetic 2^l-pole radiation fields are the two sets of irreducible representations of the rotation group of order $2l+1$, the two sets differing with respect to the reflection operator. However, instead of the usual separation of radiation into electric and magnetic multipoles, their sums and differences, called the right-hand and the left-hand circular multipoles, are used. Expressions for the multipole fields are first expressed by means of the derivatives of the fundamental spherically symmetric scalar solution $G(r)$ of the monochromatic wave equation $(\Delta+k^2)G=0$, namely, $G=e^{\pm ikr}/kr$ for in and outgoing and $\sin kr/kr$ for standing waves. The explicit expressions are then obtained in terms of the Hankel or the Bessel functions of half odd integral order, respectively, depending on whether the former or the latter expression for G is taken. The quantization is carried out in the usual way, and the angular momentum of the radiation field is calculated. The result coincides with that obtained earlier by Heitler [Proc. Cambridge Philos. Soc. **32**, 112–126 (1936)], but it is maintained that, contrary to Heitler's statement, the angular momentum resides in the wave-zone.

S. Kusaka (Northampton, Mass.).

Chakrabarty, S. K. On the convergency of the solutions of cascade equations in cosmic radiation. Bull. Calcutta Math. Soc. **36**, 9–13 (1944). [MF 11212]

A solution of the cascade equations valid for all positive values of E and t is obtained in the form

$$P(E, t)=\frac{1}{2\pi iE_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^s\left\{\sum_{r=0}^{\infty}\left(\frac{-\beta}{E}\right)^r\frac{\Gamma(s+r)}{\Gamma(s)}\psi_r(s, t)\right\}ds,$$

where $P(E, t)dE$ is the number of particles in a shower at depth t below the free surface and having energies lying between E and $E+dE$. The function $\psi_r(s, t)$ satisfies a functional differential equation. A second expression for $P(E, t)$, also valid for all positive values of E and t, is expressed as an integral involving a set of functions $\phi_r(s, t)$ which can be expressed in terms of the functions $\psi_n(s, t)$ from $n=0$ to r. The disadvantages of a solution proposed by Snyder and Serber are indicated. *H. Bateman.*

Hulthén, Lamek. A note on eigenphases and eigenfunctions of certain continuous spectra. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 8, 6 pp. (1944). [MF 10977]

The radial Schrödinger equation

$$(1)\qquad [d^2/dr^2-l(l+1)/r^2+k^2-U(r)]\psi=0$$

has solutions for the case of the continuous spectrum given asymptotically by $\psi_l(r)\sim\sin(kr-l\pi/2+\eta_l)$, analogous to to the well-known asymptotic expressions for the Bessel functions. The author calls the quantity η_l an "eigenphase." He derives formulas involving this eigenphase which are generalizations of known formulas for Bessel functions. These formulas can be applied to the approximate solution of (1) for special forms of the interaction term $U(r)$.

O. Frink (Xenia, Ohio).

Hulthén, Lamek. Variational problem for the continuous spectrum of a Schrödinger equation. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 14, no. 21, 13 pp. (1944). [MF 10978]

The usual variational methods for the approximate solution of a Schrödinger equation cannot be conveniently applied to the continuous spectrum. The author describes a new variational method, applicable to the continuous spectrum, in which the role of the eigenvalue is taken on by an "eigenphase" [see the preceding review]. The type of problem considered is that of a system of two particles with positive energy E and a general interaction term $U(r)$. Asymptotic formulas for the eigenfunctions then exist, containing a constant called the eigenphase. These formulas are analogous to the asymptotic expressions for the Bessel functions. The author shows how to set up a variational problem whose solution yields both eigenphase and eigenfunction. He applies his method to obtain numerical results in the case of the continuous spectrum of the deuteron, with an assumed interaction term of Yukawa type. *O. Frink.*

Scherrer, Willy. Über den Begriff des Atoms. I. Helvetica Phys. Acta 15, 53–73 (1942).
Scherrer, Willy. Über den Begriff des Atoms. II. Helvetica Phys. Acta 15, 476–496 (1942).

Scherrer, Willy. Über den Begriff des Atoms. III. Helvetica Phys. Acta 16, 230–234 (1943).
Scherrer, W. Über den Begriff des Atoms. IV. Comment. Math. Helv. 16, 115–145 (1944).

[Cf. the two preceding titles.] The author continues his development of a new form of relativistic wave mechanics, which resembles the de Broglie theory. Part III is a brief summary of the results of part IV. The behavior of a particle of mass m in a force-free field is to be described by the wave equation: (1) $\nabla^2 u - \partial^2 u/c^2\partial t^2 = a^2 u$, where $a = 2\pi mc/h$. The integral of u^2 over an allowable region of space-time is interpreted as the relative probability of encountering the particle in that region. The restriction of particle velocities to values less than c is incorporated into the boundary conditions on u; the function u for a particle appearing at a given place and time is to be defined only over the light-cone with the corresponding vertex.

The mathematically interesting part of the paper is the author's method of solving (1). He separates the variables in the coordinate system $(\tau, \sigma, \theta, \phi)$, where $x = r \sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$, $Act = \tau\sigma$ and $Ar = (A^2 - \tau^2)^{\frac{1}{2}}(A^2 - \sigma^2)^{\frac{1}{2}}$. This is a combination of spherical and spheroidal coordinates. Setting $u = T(\tau)S(\sigma)P(\theta, \phi)$, it is found that P is an ordinary spherical harmonic, while both T and S satisfy a modified form of the Mathieu equation. The author expands T and S in series of four-dimensional spherical harmonics. The results are used to describe the behavior of a particle which appears at the origin at the two times $ct = \pm A$, the so-called two-point problem.

Some conclusions concerning the asymptotic behavior for large r and t are derived. For example, at great distances the frequency and wave-length approach those of the de Broglie theory. *O. Frink* (Xenia, Ohio).

Ginsburg, V. and Smorodinskij, J. On wave equations for particles with variable spin. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 13, 274–276 (1943). (Russian) [MF 11219]

Davydov, A. S. Wave equation of a particle with a 3/2 spin in the absence of the field. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 13, 313–319 (1943). (Russian) [MF 11223]

Tonnelat, Marie-Antoinette. Sur les équations d'ondes des particules à spin en présence d'un champ extérieur. C. R. Acad. Sci. Paris 217, 385–388 (1943). [MF 11107]

Petiau, Gérard. Sur les ondes planes de la théorie des corpuscules de spins quelconques. C. R. Acad. Sci. Paris 216, 196–198 (1943). [MF 10953]

Bunge, Mario. The total spin of a mass-system of two particles. Revista Union Mat. Argentina 10, 13–14 (1944). (Spanish) [MF 11028]

Nikolskij, K. Relativistic formulation of quantum interaction. I. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 13, 277–283 (1943). (Russian) [MF 11220]

Kronig, R. und Korringa, J. Zur Theorie der Bremsung schneller geladener Teilchen in metallischen Leitern. Physica 10, 406–418 (1943). [MF 11421]

This paper gives a new theory of the contribution of the conduction electrons to the retarding force acting on a charged particle traversing a metal. The earliest theories predicted an infinite value for the energy lost by the particle per unit of distance traversed. Subsequent theories, in which account was taken of the energy exchanges between the conduction electrons and the positive ions, led to finite values. The present authors show that a finite value is obtained, even when the above-mentioned energy exchanges are neglected, if the interaction between the conduction electrons themselves is taken into account more correctly. The theory is phenomenological, the conduction electron gas being treated as a continuous fluid, having internal friction and a frictional interaction with the substratum of positive charge. The theory indicates that the frictional interaction with the positive substratum is in fact of negligible effect in comparison with the internal friction of the conduction electron fluid. *L. A. MacColl.*

Fierz, M. Zur Theorie magnetisch geladener Teilchen. Helvetica Phys. Acta 17, 27–34 (1944).

It was proved by Dirac [Proc. Roy. Soc. London, Ser. A. **133**, 60–72 (1931)] that single magnetic poles are compatible with the principles of quantum mechanics if their magnetic charge p fulfills the condition that $\mu = ep/\hbar c$ (e elementary electric charge, $2\hbar$ quantum of action, c velocity of light) is either an integer or a half-integer. In the present paper a new proof for Dirac's theorem is given. Considering the motion of an electric charge e in the field of a magnetic pole p, it is first shown that classically ep/c is the amount of an additional angular momentum (besides the orbital angular momentum) parallel to the line joining the magnetic and the electric charge. The second step of the proof consists in computing the wave mechanical operators corresponding to the components of the total angular momentum and applying Pauli's criterion [same Acta 12, 147–168 (1939)], according to which successive applications of this operator to the admissible eigensolutions of the Schrödinger equation leads to regular solutions. This is here the case only if μ is either an integer or a half-integer. *W. Pauli* (Princeton, N. J.).

Fierz, Markus. Über die Wechselwirkung zweier Nukleonen in der Mesontheorie. Helvetica Phys. Acta 17, 181–194 (1944).

In the case of strong coupling, the symmetrical meson theory (either with pseudoscalar or vector or mixed fields) leads to an isobar energy of one heavy particle (nucleon) of the form $(\epsilon/2)\sum_{k=1}^{3} d_k^2$ and an interaction energy of two nucleons of the form $V(r)\cdot\Omega$, where $\vec{d}$ is the spin momentum and

$$\Omega=\sum_{i,k=1}^{3} x_{ik}x_{ik}'$$

with x_{ik} and x_{ik}' orthogonal matrices for each of the nucleons, respectively. The x_{ik} can be expressed by three Euler angles and the d_k by differential operators with respect to them. This simple form of the interaction energy holds only if the tensor force is neglected. The eigenvalues of $\sum_k d_k^2$ and of $\sum_k d_k'^2$ for the second nucleon are given by $j(j+1)$ and $j'(j'+1)$ with j and j' half odd integers. Besides there exist the isotopic spin components h_k, h_k' for which $\sum_k h_k^2=\sum_k d_k^2$, $\sum_k h_k'^2=\sum_k d_k'^2$. According to the well-known composition of two representations of the rotation group, the operators $\sum_k(d_k+d_k')^2$ and $\sum_k(h_k+h_k')^2$ have the eigenvalues $J(J+1)$ and $K(K+1)$, respectively, with J, K integers in the interval $|j-j'|\leqq J\leqq j+j'$, $|j-j'|\leqq k\leqq j+j'$. In the present paper the matrix elements $J, K, j, j'|\Omega|J, K, J, J'$ are explicitly computed by algebraic methods. (Note that Ω is diagonal with respect to J, K.)

W. Pauli (Princeton, N. J.).

Fierz, M. und Wentzel, G. Zum Deuteronproblem. I. Helvetica Phys. Acta 17, 215–232 (1944).

The deuteron problem (two-nucleon problem) in the symmetrical meson theory is treated in detail for the case of strong coupling [compare also R. Serber and S. M. Dancoff, Phys. Rev. (2) 63, 143–161 (1943); these Rev. 4, 207 and W. Pauli and S. Kusaka, Phys. Rev. (2) 63, 400–416 (1943); these Rev. 5, 55] with neglect of the tensor force. The interaction is given by $V(r)\cdot\Omega$, where Ω is defined in the paper reviewed above, and concerning the function $V(r)$ of the distance r between the nucleons it is only supposed that $V(r)>0$. The wave-mechanical problem can be considered as describing two spherical tops moving around their center of gravity under the influence of their interaction energy. Two approximation methods for solving this problem are discussed: (1) the adiabatic method: the problem is first solved with a fixed r as parameter, and transitions between the states corresponding to fixed r due to the motion of the particles are treated later by a perturbation method; (2) the variation method: $V(r)$ is here replaced by an average value $\bar{V}$ over r. The dimensionless parameter of the problem is V/ϵ. The two limiting cases of very small and very large values of V/ϵ can be solved, and a one-to-one correspondence between the stationary states of these two limiting cases can be established.

In the present paper only the states with $J=K=0$ are considered. An important result is that the adiabatic method leads, for the states $J=K=0$, to a deepest eigenvalue $W(r)$ of the energy which is positive for large r (V/ϵ small), negative for small r (V/ϵ large). *W. Pauli.*

Wentzel, G. Zum Deuteronproblem. II. Helvetica Phys. Acta 17, 252–272 (1944).

The treatment of the deuteron problem for the strong coupling case which was given in Part I [cf. the preceding review] for states with $J=K=0$ is extended to states with either $J=0$, $K\neq 0$ or $J\neq 0$, $K=0$ (as long as the tensor force is neglected the energy is symmetrical in the quantum numbers J of the resulting spin, and K of the resulting isotopic spin.) Use is made of the evaluation of the matrix elements of Ω by Fierz [see the second review above]. In the case $V/\epsilon\ll 1$ the excitation of the isobars can be treated as a small perturbation while in the opposite case $V/\epsilon\gg 1$ one has in first approximation small harmonic oscillations of the angles around equilibrium positions. For the ground state of the deuteron $J=1$, $K=0$ and it is satisfactory that for this state the adiabatic method [see the preceding review] gives a negative energy value for all r (attraction). Besides J and K the stationary states of the two nucleon system are characterized by a quantum number n which is even (odd) if $J+K$ is odd (even). States with the same value of n but different values of J or K (one of them being assumed to be zero) are "isobaric" in the sense that their energy difference is in good approximation independent of the value of V/ϵ [see the preceding review]. *W. Pauli.*

Lifshitz, I. M. Optical behaviour of non-ideal crystal lattices in the infra-red. I. Acad. Sci. USSR. J. Phys. 7, 215–228 (1943). [MF 10999]

The first problem is to find the electric moment P for a crystal excited by a field $Ee^{i\omega t}$. If, under the influence of this field, an atom occupying the sth place in the kth cell has a displacement $b_k^s(t)=b_k^s e^{i\omega t}$ and a charge e_k^s, then

$$P=\sum_{k,s} e_k^s b_k^s.$$

For an ideal lattice e_k^s is the same for all values of k and there is a simplification. Also, when only the quadratic terms in the expansion of the lattice energy in powers of the displacements are used, the equations of motion are differential equations of the second order in the set of quantities $b_k^{s,x}(t)$ with coupling coefficients $B_{k-k'}^{sx,s'x'}$ which depend only on the differences $k-k'$. The corresponding coupling coefficients depend on k and k' in a different way when the lattice is not ideal and the masses multiplying the second derivatives are quantities m_k^s which depend on both k and s. When only small deviations are considered and a single symbol σ is used to denote a combination of the two indices, the equation of perturbations may be reduced to a matrix form

$$(A-\omega^2+\Lambda)u=\phi+\delta,$$

where $u^\sigma=b_k^\sigma\sqrt{m_s}$ and A is a matrix with elements $A_{k-k'}^{\sigma\sigma'}$ of the form

$$A_{k-k'}^{\sigma\sigma'}=(m_s m_{s'})^{-\frac{1}{2}}B_{k-k'}^{\sigma\sigma'}.$$

The corresponding equation for the ideal lattice is $(A-\omega^2)u=\phi$, so that Λ is the perturbation operator or matrix and δ is the correction term on the right hand side. Moreover, Λ has one form for a mixture of isotopes of the components of an ionic lattice and another form for mixed crystals with a small concentration of one component. For the first case Λ is easily defined and it is observed that each of its elements is small compared with the corresponding elements of A. The properties of the operator A are discussed and $A_n^{\sigma\sigma'}$ is regarded as the coefficient of $\exp(2\pi in\theta)$ in a Fourier series of a Hermitian matrix $a^{\sigma\sigma'}(\theta)$. The solution of the unperturbed problem is given and the equation for the eigenvalues has roots which are interpreted as giving the acoustic and optical branches of the crystal.

H. Bateman (Pasadena, Calif.).

AUTHOR INDEX

PRINTED IN THE UNITED STATES OF AMERICA